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The helicoidal modeling in computational finite elasticity. Part I: Variational formulation

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Abstract

The finite elasticity mechanics of continua capable of a polar description is formulated by an alternative modeling to keeping position and orientation as uncoupled fields. The rototranslation between two material particles can be described by a single, complex tensorial quantity, which is recognized to be orthogonal. Its linearization gives the characteristic curvature and differential vectors underlying the helicoidal modeling in both the sense of the body geometric description and the evolution of a deforming body. After due introduction to dual tensors and rototranslations, the polar description of the continuum is addressed, with particular care to mixed differentiations of the rototranslation field. Then, a thorough variational framework is established for the most general polar continuum under hyperelasticity hypothesis, and the three-field, two-field and one-field principles are drawn and linearized. The proposed modeling is expected to be profitably exploited in non-linear finite element analyses of solids undergoing finite displacements, rotations and strains.

Keywords: Finite elasticity; Variational formulations; Polar medium; Finite rotations; Dual algebra

1. Introduction

The use of three-dimensional, finite rotations in solid modeling is propagating in the field of continuum mechanics, as proved by several papers issued during the last decade and concerning either the theoretical background on appropriate variational formulations or numerical applications by the finite element method. Besides a number of works addressing the local behavior of materials that can support couple-stresses (e.g. Le and Stumpf, 1998; Nikitin and Zubov, 1998; Grekova and Zhilin, 2001), the interest is often focused on mechanical formulations with aim to improve the response in geometrically non-linear problems (e.g. Simo et al., 1992; Kožar and Ibrahimbegović, 1995; Atluri and Cazzani, 1995; Merlini, 1997; Wisniewski, 1998). Progress in the latter field has often been furthered by advances in shell theories and

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elements (see e.g.—with no claims to be comprehensive—Büchter and Ramm, 1992; Wriggers and Gruttmann, 1993; Ibrahimbegović, 1994; Sansour and Bednarczyk, 1995; Wisniewski, 1998; Li and Zhan, 2000; Wang and Thierauf, 2001; Ibrahimbegović et al., 2001) and in space-curved beam formulations (e.g. Borri and Bottasso, 1994a,b; Ibrahimbegović et al., 1995; Crisfield and Jelenić, 1999; Jelenić and Crisfield, 1999; Atluri et al., 2001). In almost all the works quoted, the rotation field is understood as an independent field to be considered almost in the same way as the main kinematical unknown field, the displacement, is considered. However, it must be observed that such two fields are deeply different in character, the displacement belonging to the standard Euclidean vector space and the rotation to the group of special orthogonal transformations. It follows that the treatment of the two fields is quite different: for instance, displacements commute and compose additively, while rotations do not commute and compose multiplicatively. Owing to this fact, it was soon realized that a multiplicative updating of the rotations was necessary in order to avoid singularities during an incremental solution process, as recognized by several of the abovementioned authors and recently analyzed by Betsch et al. (1998). Despite this fact and oddly enough, however, the need of a multiplicative interpolation of the rotation field was overlooked and standard additive interpolating schemes over the element domain have been exploited by almost all authors but Borri and Bottasso (1994a,b), Crisfield and Jelenić (1999) and Jelenić and Crisfield (1999) in the quite simple, one-coordinate case of beams.

According to the commonly accepted approach, the displacement and rotation fields are assumed as uncoupled. However, a different point of view may be conceived where displacements and rotations are understood as components of a unique, integral field. The idea is better explained with reference to the elementary motion of a frame, see Fig. 1. In the classical view, the simplest direct motion between two oriented frames 1 and 2 is described by a rotation occurring synchronously with the displacement along a straight path. This means that rotation and displacement are controlled separately by the extreme orientations and positions, respectively, and leads to a description where the path is determined by positions with no contribution at all from the orientations. In contrast, in the alternative approach the frame moves along a curved path whose curvature is dictated by the extreme orientations themselves, and describes a helix in space. This kind of parameterization of motion is the subject of recent work by Borri and coworkers in the field of multibody dynamics (Borri et al., 1998; Borri et al., 2000). Application of this concept to a deformable three-dimensional body leads to a novel scheme of continuum modeling that will be called helicoidal modeling. It will be shown that both position and orientation of an oriented particle within a continuum can be identified by a single tensorial quantity, and its motion can be described by a single tensorial quantity referred to as rototranslation; this tensorial quantity is seen to be orthogonal, and to inherit all the properties of the rotation tensor. It is worth noting that the proposed modeling concerns the curved path between any two oriented frames, hence it is relevant both to the evolution of one particle

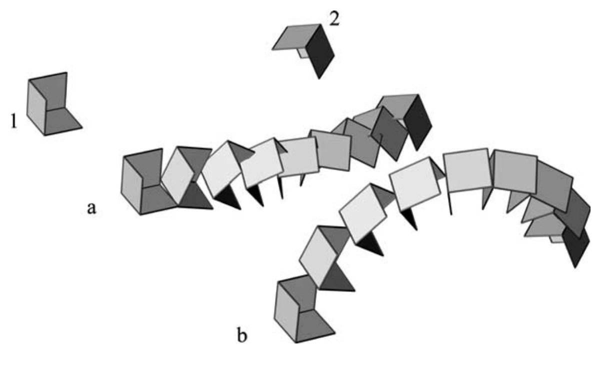

Fig. 1. Frame motion from 1 to 2: (a) classical uncoupled modeling and (b) helicoidal modeling.

along the deformation history, and the body geometry as described by different particles along the material coordinates; so, the particle orientations do contribute to define both the particle trajectories and the material lines across a body. Resorting to this helicoidal modeling can be advantageous in the context of approximate solutions, where a helicoidal updating technique during the iterative solution process together with helicoidally interpolated elements are expected to improve in quality the analysis of geometrically non-linear problems.

Two tasks are attempted in this paper. On one hand, the paper aims to establish the polar kinematics of a deformable continuum on the basis of the rototranslation of material particles, and to introduce the concept of helicoidal modeling. To this aim, the objects undergoing rototranslations are characterized in Section 2, on use of the powerful rules of the algebra of dual numbers (Angeles, 1998); then, the rototranslation is defined as a magnitude-preserving transformation and the relevant properties are discussed. The differentiations up to third-order of the rototranslation are analyzed in Section 3, and the relevant characteristic differential vectorial quantities (the differential helices) are identified. This work is preliminary to Section 4, which is devoted to the polar description of the continuum (Cosserat and Cosserat, 1909; Eringen and Kafadar, 1976) by using a rototranslation field. The tangent space of such a configuration field is characterized by a differential helix, and this fact defines the helicoidal modeling. In particular, the curvature within the continuum is determined, and the kinematical measure of strain is identified as the appropriate difference of curvature from the initial to the deformed configuration.

On the other hand, this paper aims to establish the fundamental principles of solid mechanics for finite elasticity, written in the light of the proposed modeling. With reference just to a hyperelastic behavior, the equations of the continuum are given and brought into a thorough variational framework; then, the main variational principles in elastostatics are derived and consistently linearized. This task is accomplished in Section 5 for a polar medium (Toupin, 1962, 1964; Eringen and Kafadar, 1976), capable of withstanding angular strains by opposing couple-stresses and thus exploiting better the helicoidal modeling. However, the proposed general theory of a polar continuum applies as well to other simpler materials without couple-stresses, once appropriate restrictions are postulated for the strain-energy function.

This article is part of three companion papers. Part II deals with the multiplicative interpolation of an orthogonal field to be used in finite-element approximations. In Part III, the mechanics of the polar continuum is particularized for the common case of non-polar medium, and by taking advantage of the helicoidal modeling, suitable finite elements are formulated for the analysis of geometrically non-linear problems with large rotations.

2. Introducing rototranslations

Rototranslations are introduced from a pure geometrical standpoint. The objects undergoing rototranslations are first characterized: just like rotations apply to vectors, so rototranslations apply to more complex geometrical entities such as the pair vector and moment. The basic properties of such geometrical entities are discussed, and the rototranslation is then defined as a magnitude-preserving transformation of such objects. It is operated by a special tensor, called the rototranslation tensor, which is shown to inherit all the properties of the rotation tensor. In particular, it is endowed with the orthogonality property and allows the form of the exponential expansion of a skew-symmetric argument.

2.1. Dual vectors and tensors

First, consider an *applied vector* at a point in space. The pair of the vector itself and its moment with respect to an arbitrarily chosen *pole* is a geometrical entity representative of the applied vector—symbolically $(\boldsymbol{v}, \underset{O}{\boldsymbol{d}} \times \boldsymbol{v})$, with $\boldsymbol{v}$ the vector and $\underset{O}{\boldsymbol{d}}$ the relative position of the application point from pole O. Indeed, the same pair is representative of the same vector applied at any other point $\underset{O}{\boldsymbol{d}}' = \underset{O}{\boldsymbol{d}} + pv^{-1}\boldsymbol{v}$ (p any real length, v the magnitude of $\boldsymbol{v}$) lying on the straight line aligned with the vector itself; this line is called the *axode* of the applied vector. Then, consider a pure *couple* (of vectors). In such case, the location of the couple is immaterial and the same moment is anyway measured with respect to any chosen pole. Again, the pair $(\mathbf{0}, \boldsymbol{c})$, with a null vector, may represent a couple $\boldsymbol{c}$. Finally, consider an *applied vector-and-couple* at a point and its representation through the most general geometrical pair $(\boldsymbol{v}, \underset{O}{\boldsymbol{m}})$ made of vector $\boldsymbol{v}$ and moment $\underset{O}{\boldsymbol{m}} = \boldsymbol{c} + \underset{O}{\boldsymbol{d}} \times \boldsymbol{v}$ including a possible couple $\boldsymbol{c}$. Notice that actually, the same pair is representative of any other applied vector-and-couple based on the same pole, with possibly different application point $\underset{O}{\boldsymbol{d}}$ and couple $\boldsymbol{c}$, still preserving the sum $\boldsymbol{c} + \underset{O}{\boldsymbol{d}} \times \boldsymbol{v}$. Nevertheless, a geometrical pair like $(\boldsymbol{v}, \underset{O}{\boldsymbol{m}})$ is a good prototype of objects undergoing rototranslations.

Various names have been proposed in literature for the pair $(\boldsymbol{v}, \underset{O}{\boldsymbol{m}})$. The term *wrench*, as used by Hestenes (1986) with reference to a complex force acting on a rigid body, is appropriate, however a more terse geometric terminology, free of any mechanical implications, is preferable in an introductory context. It is also worth noting that the pair in discussion is made of two distinct vectors having different physical dimensions, the moment dimension being greater than the vector dimension by a length. This fact can cause difficulties when settling the relevant algebra: Borri et al. (1998, 2000) resorted to a perhaps cumbersome matrix arrangement involving vectorial and tensorial entities defined in a six-dimensional space. We prefer to follow Angeles (1998) who promoted using the dual algebra in the kinematics of mechanical systems. The pair representing an applied vector-and-couple at a point is simply called a *dual vector*, a vectorial entity made of two distinct vectors fit for such kind of objects, and the relevant algebra is a straightforward extension of the commonly used tensor algebra to encompass the powerful rules of the algebra of dual numbers (Bottema and Roth, 1979). The reader must be aware that the term dual vector, as used here and throughout the paper, has nothing to do with the duality between the spaces of covariant and contravariant vectors or between the tangent and co-tangent spaces as commonly understood in tensor algebra and Hamiltonian mechanics (Bowen and Wang, 1976; Arnold, 1989).

Box 1. Basic rules of dual algebra

Dual numbers were first proposed by Clifford (1873), and then applied to kinematics since Kotel'nikov (1895) and Study (1903). A *dual number* is made of two distinct real numbers, a *primal* part p and a *dual* part d, and is defined as the sum $p + \varepsilon d$, with ε the *dual unity* endowed with the properties $\varepsilon \neq 0$ and $\varepsilon^2 = \varepsilon^3 = \cdots = 0$ (Bottema and Roth, 1979). A dual number missing of the primal part is said a *pure* dual number, while in a *proper* dual number neither primal nor dual parts are null. The algebra of dual numbers is easily written owing to the vanishing product of the dual unity by itself; see Box 1 for the most basic rules as given by Angeles (1998). In his work, Angeles also applied these rules to vectors and matrices: some useful formulae for dual tensors are gathered in Box 2.

$$
\begin{aligned}
a &= p + \varepsilon d \\
b &= q + \varepsilon e \\
a = b &\Longleftrightarrow \begin{cases} p = q \\ d = e \end{cases} \\
a + b &= (p + q) + \varepsilon (d + e) \\
ab &= pq + \varepsilon (pe + dq) \\
\frac{a}{b} &= \frac{p}{q} - \varepsilon \frac{pe - dq}{q^2} \quad (q \neq 0)
\end{aligned}
\qquad
\begin{aligned}
\frac{1}{a} &= \frac{1}{p} - \varepsilon \frac{d}{p^2} \\
a^0 &= 1 \\
\sqrt{a} &= \sqrt{p} + \varepsilon \frac{d}{2\sqrt{p}} \\
a^2 &= p^2 + \varepsilon 2pd \\
a^n &= p^n + \varepsilon n p^{n-1} d
\end{aligned}
\qquad
\begin{aligned}
\exp a &= \exp p + \varepsilon d \exp p \\
\sin a &= \sin p + \varepsilon d \cos p \\
\cos a &= \cos p - \varepsilon d \sin p \\
\tan a &= \tan p + \varepsilon \frac{d}{\cos^2 p}
\end{aligned}
$$

Box 2. Formulae for dual tensors

$\boldsymbol{A} = \boldsymbol{P} + \varepsilon \boldsymbol{D}$	$\boldsymbol{A}^{-1} = \boldsymbol{P}^{-1} - \varepsilon \boldsymbol{P}^{-1}\boldsymbol{D}\boldsymbol{P}^{-1}$
$\boldsymbol{B} = \boldsymbol{Q} + \varepsilon \boldsymbol{E}$	$\boldsymbol{A}^{0} = \boldsymbol{I}$
$\boldsymbol{A} = \boldsymbol{B} \Longleftrightarrow \begin{cases} \boldsymbol{P} = \boldsymbol{Q} \\ \boldsymbol{D} = \boldsymbol{E} \end{cases}$	$\boldsymbol{A}^{2} = \boldsymbol{P}^{2} + \varepsilon(\boldsymbol{P}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{P})$ $\boldsymbol{A}^{n} = \boldsymbol{P}^{n} + \varepsilon \sum_{m=1}^{n} \boldsymbol{P}^{n-m}\boldsymbol{D}\boldsymbol{P}^{m-1}$
$\boldsymbol{A} + \boldsymbol{B} = (\boldsymbol{P} + \boldsymbol{Q}) + \varepsilon(\boldsymbol{D} + \boldsymbol{E})$	$\exp \boldsymbol{A} = \exp \boldsymbol{P} + \varepsilon \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{(n+m+1)!} \boldsymbol{P}^{n}\boldsymbol{D}\boldsymbol{P}^{m}$
$\boldsymbol{A}\boldsymbol{B} = \boldsymbol{P}\boldsymbol{Q} + \varepsilon(\boldsymbol{P}\boldsymbol{E} + \boldsymbol{D}\boldsymbol{Q})$	$\mathrm{tr}\boldsymbol{A} = \mathrm{tr}\boldsymbol{P} + \varepsilon\,\mathrm{tr}\boldsymbol{D}$
$\boldsymbol{A}\boldsymbol{B}^{-1} = \boldsymbol{P}\boldsymbol{Q}^{-1} - \varepsilon(\boldsymbol{P}\boldsymbol{Q}^{-1}\boldsymbol{E}\boldsymbol{Q}^{-1} - \boldsymbol{D}\boldsymbol{Q}^{-1})$	$\mathrm{tr}^2\boldsymbol{A} - \mathrm{tr}\boldsymbol{A}^2 = (\mathrm{tr}^2\boldsymbol{P} - \mathrm{tr}\boldsymbol{P}^2) + \varepsilon(2\mathrm{tr}\boldsymbol{P}\mathrm{tr}\boldsymbol{D} - \mathrm{tr}(\boldsymbol{P}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{P}))$
$\boldsymbol{B}^{-1}\boldsymbol{A} = \boldsymbol{Q}^{-1}\boldsymbol{P} - \varepsilon(\boldsymbol{Q}^{-1}\boldsymbol{E}\boldsymbol{Q}^{-1}\boldsymbol{P} - \boldsymbol{Q}^{-1}\boldsymbol{D})$	$\det \boldsymbol{A} = \det \boldsymbol{P}(1 + \varepsilon\,\mathrm{tr}(\boldsymbol{P}^{-1}\boldsymbol{D}))$

Of course, physical dimensions must be respected when defining the operations given in Box 2, and it can be seen that this condition is fulfilled when dual parts differ from the primal parts by a unique, common dimension. All the dual entities we are interested in have one part dimensionally greater than the other by a length. So, an applied vector-and-couple at a point is conveniently represented by a *dual vector* like

$$\underset{O}{\boldsymbol{w}} = \boldsymbol{v} + \varepsilon \underset{O}{\boldsymbol{m}}.$$

With reference to two reciprocal frames of covariant and contravariant base vectors $\boldsymbol{g}_j$ and $\boldsymbol{g}^j$ ($j = 1, 2, 3$), in general non-orthonormal, the same dual vector can be seen as a vector with dual numbers as components, $\underset{O}{\boldsymbol{w}} = \underset{O}{w_j}\boldsymbol{g}^j = (v_j + \varepsilon \underset{O}{m_j})\boldsymbol{g}^j$, or alternatively $\underset{O}{\boldsymbol{w}} = \underset{O}{w^j}\boldsymbol{g}_j = (v^j + \varepsilon \underset{O}{m^j})\boldsymbol{g}_j$. This allows applying the well established tensor algebra to the new geometrical object, by just honoring the rules on dual numbers for the vector components. In particular, dual vectors based on a unique pole can be superposed like standard vectors to obtain a *resultant* dual vector made of the resultant vector and the resultant moment with respect to the pole. Equipollent applied vector-and-couples (Hestenes, 1986) are obviously represented by the same dual vector, so different vector-and-couple systems having the same resultant dual vector are said to be *equipollent*; a vector-and-couple system with a pure dual resultant is equipollent to a pure couple independent of any pole.

The dyadic representation of tensors helps extending this formalism to higher order geometrical objects. A second-order tensor can be written in dyadic form as $\boldsymbol{V} = \boldsymbol{V}_j \otimes \boldsymbol{g}^j = \boldsymbol{V}^j \otimes \boldsymbol{g}_j$, hence it corresponds to either a triad of vectors $\boldsymbol{V}_j = \boldsymbol{V} \cdot \boldsymbol{g}_j$ or $\boldsymbol{V}^j = \boldsymbol{V} \cdot \boldsymbol{g}^j$, namely the covariant or contravariant (right) component vectors of tensor $\boldsymbol{V}$, respectively. Consider an *applied tensor* at a point in the space. The three pairs $(\boldsymbol{V}_j, \underset{O}{\boldsymbol{d}} \times \boldsymbol{V}_j)$, or alternatively $(\boldsymbol{V}^j, \underset{O}{\boldsymbol{d}} \times \boldsymbol{V}^j)$, made of the component vectors and their moments with respect to pole O, are representative of the applied tensor, and so are representative the relevant (right) dyadic compositions $(\boldsymbol{V}_j \otimes \boldsymbol{g}^j, \underset{O}{\boldsymbol{d}} \times \boldsymbol{V}_j \otimes \boldsymbol{g}^j)$ and $(\boldsymbol{V}^j \otimes \boldsymbol{g}_j, \underset{O}{\boldsymbol{d}} \times \boldsymbol{V}^j \otimes \boldsymbol{g}_j)$, i.e. the tensorial pair $(\boldsymbol{V}, \underset{O}{\boldsymbol{d}} \times \boldsymbol{V})$. It is worth noting that the axodes of vectors $\boldsymbol{V}_j$ (or $\boldsymbol{V}^j$) converge in the application point; therefore, the pair $(\boldsymbol{V}, \underset{O}{\boldsymbol{d}} \times \boldsymbol{V})$—unlike an applied vector $(\boldsymbol{v}, \underset{O}{\boldsymbol{d}} \times \boldsymbol{v})$—is able to fully determine both in value and location a non-singular applied tensor $\boldsymbol{V}$. Then, a pure *couple-tensor* $\boldsymbol{C}$ could be considered, and more generally an *applied tensor-and-couple-tensor* at a point, made of tensor $\boldsymbol{V}$ and a moment-tensor $\underset{O}{\boldsymbol{M}} = \boldsymbol{C} + \underset{O}{\boldsymbol{d}} \times \boldsymbol{V}$ with respect to pole O. The applied tensor-and-couple-tensor is conveniently represented by the *dual tensor*

$$\underset{O}{\boldsymbol{W}} = \boldsymbol{V} + \varepsilon \underset{O}{\boldsymbol{M}}.$$

The recursive extension to higher order tensors broadens the family of geometrical objects of this kind. We use third-order tensors in this paper, and we represent an applied third-order tensor-and-couple-tensor at a point as a dual tensor like $\underset{O}{\mathscr{W}} = \mathscr{V} + \varepsilon \underset{O}{\mathscr{M}}$ with $\underset{O}{\mathscr{M}} = \mathscr{C} + \underset{O}{\boldsymbol{d}} \times \mathscr{V}$, where for instance $\mathscr{V} = \mathscr{V}_j \otimes \boldsymbol{g}^j = \mathscr{V}^j \otimes \boldsymbol{g}_j$, etc.

It is worth noting that the algebra of dual tensors is a straightforward extension of the well established tensor algebra. This fact constitutes the power of this representation, in the sense that the simple yet effective rules of the algebra of dual numbers can be applied in a way transparent to tensor algebra. Thus, all the properties of real tensors keep valid for dual tensors. In particular, it will be shown in next sections that the magnitude of dual vectors and the determinant of second-order dual tensors are unequivocally defined and result in pole-independent scalar invariants. The advantage of this representation against a matrix arrangement as used by Borri et al. (1998, 2000) is strongly felt in three-dimensional continuum mechanics when working with field derivative operators (gradient and divergence).

2.2. Pole basing

It is worth noting that the foregoing geometrical objects do depend on the point chosen as a pole. When the pole is distinct from the application point itself, they will be also referred to as *pole-based* dual vectors and tensors. As a particular case, however, consider making the pole coincident with the application point: such objects would reduce to $\boldsymbol{w} = \boldsymbol{v} + \varepsilon \boldsymbol{c}$, $\boldsymbol{W} = \boldsymbol{V} + \varepsilon \boldsymbol{C}$ and $\mathscr{W} = \mathscr{V} + \varepsilon \mathscr{C}$, and in this case they will be referred to as *self-based* dual vectors and tensors.

Pole-based dual vectors and tensors can always be obtained from the relevant self-based ones through linear transforms, e.g. $\underset{O}{\boldsymbol{w}} = \underset{O}{\boldsymbol{D}} \boldsymbol{w}$, $\underset{O}{\boldsymbol{W}} = \underset{O}{\boldsymbol{D}} \boldsymbol{W}$ and $\underset{O}{\mathscr{W}} = \underset{O}{\boldsymbol{D}} \mathscr{W}$. The transform operator is defined as the dual tensor

$$\underset{O}{\boldsymbol{D}} = \boldsymbol{I} + \varepsilon \underset{O}{\boldsymbol{d}} \times, \tag{1}$$

made of the identity $\boldsymbol{I}$ as primal part and of a skew-symmetric tensor $\underset{O}{\boldsymbol{d}} \times$ as dual part. The latter is given a notation based on the *axial vector* $\underset{O}{\boldsymbol{d}}$ of the tensor itself. We shall usually denote a skew-symmetric tensor by the relevant axial vector followed by the cross-product symbol, as it corresponds to the linear vector operator allowing to write the cross product between two vectors as $\boldsymbol{a} \times \boldsymbol{b} = (\boldsymbol{a} \times) \boldsymbol{b}$. Using the algebraic rules of dual numbers, the pole-basing operations are easily checked, for example $\underset{O}{\boldsymbol{w}} = \underset{O}{\boldsymbol{D}} \boldsymbol{w} = (\boldsymbol{I} + \varepsilon \underset{O}{\boldsymbol{d}} \times)(\boldsymbol{v} + \varepsilon \boldsymbol{c}) = \boldsymbol{v} + \varepsilon (\boldsymbol{c} + \underset{O}{\boldsymbol{d}} \times \boldsymbol{v})$. Tensor $\underset{O}{\boldsymbol{D}}$ is sometimes called the transport operator, or the *arm* operator. Its inverse $\underset{O}{\boldsymbol{D}}^{-1} = \boldsymbol{I} - \varepsilon \underset{O}{\boldsymbol{d}} \times$ is an arm operator itself built on the inverse distance $-\underset{O}{\boldsymbol{d}}$, and coincides with its transpose $\underset{O}{\boldsymbol{D}}^{\mathrm{T}}$. (The transpose of a dual tensor is the dual tensor made of the transpose of the primal and dual parts, respectively.) So, the arm operator is a proper orthogonal dual tensor, $\underset{O}{\boldsymbol{D}} \underset{O}{\boldsymbol{D}}^{\mathrm{T}} = \underset{O}{\boldsymbol{D}}^{\mathrm{T}} \underset{O}{\boldsymbol{D}} = \boldsymbol{I}$ and $\det \underset{O}{\boldsymbol{D}} = 1$.

Changing the base point involves another linear transform between differently pole-based vectors and tensors, for instance $\underset{Q}{\boldsymbol{w}} = \underset{Q}{\boldsymbol{D}_O} \underset{O}{\boldsymbol{w}}$, $\underset{Q}{\boldsymbol{W}} = \underset{Q}{\boldsymbol{D}_O} \underset{O}{\boldsymbol{W}}$ and $\underset{Q}{\mathscr{W}} = \underset{Q}{\boldsymbol{D}_O} \underset{O}{\mathscr{W}}$. The transform operator $\underset{Q}{\boldsymbol{D}_O} = \underset{Q}{\boldsymbol{D}} \underset{O}{\boldsymbol{D}}^{-1} = \boldsymbol{I} + \varepsilon \underset{Q}{\boldsymbol{d}_O} \times$ is the arm of pole O from the new pole Q, built with the distance vector $\underset{Q}{\boldsymbol{d}_O} = \underset{Q}{\boldsymbol{d}} - \underset{O}{\boldsymbol{d}}$.

Most classical formulations in continuum mechanics stand on self-based dual tensors and avoid fixing any base point. The advantage of using pole-based instead of self-based dual tensors is that dual tensors based on a unique pole are straightforwardly composed when drawing the effect of multiple applied tensor-and-couple-tensor systems at different points of the same body. Moreover, the pole-based representation is endowed with a property of geometric invariance worthy to exploit in numeric processes on structured continua like shells and beams, as recently discussed by Bottasso et al. (2002). Dual tensors based on a unique pole will be referred to as *consistent* dual tensors. In our work, we are mostly concerned with consistent pole-based dual tensors and occasionally with the relevant self-based ones.

2.3. Magnitude of dual vectors

We first recall a classical theorem in mechanics (see for instance Levi-Civita and Amaldi, 1922). *Any applied vector-and-couple pair in space is equipollent to a co-axial applied vector-and-couple pair*. A proof is given as follows.

Consider a pole-based proper dual vector $\underset{O}{\boldsymbol{w}} = \boldsymbol{v} + \varepsilon \underset{O}{\boldsymbol{m}}$ representing an applied vector-and-couple pair $(\boldsymbol{v}, \boldsymbol{c})$ at a point, with moment $\underset{O}{\boldsymbol{m}} = \boldsymbol{c} + \underset{O}{\boldsymbol{d}} \times \boldsymbol{v}$ with respect to pole O. Under a change of pole from O to another point A, the same applied vector-and-couple pair $(\boldsymbol{v}, \boldsymbol{c})$ would be represented by the A-based proper dual vector $\underset{A}{\boldsymbol{w}} = \boldsymbol{v} + \varepsilon \underset{A}{\boldsymbol{m}}$, with moment $\underset{A}{\boldsymbol{m}} = \boldsymbol{c} + \underset{A}{\boldsymbol{d}} \times \boldsymbol{v}$. Moment $\underset{A}{\boldsymbol{m}}$ differs from $\underset{O}{\boldsymbol{m}}$ according to the transport formula

$$\underset{A}{\boldsymbol{m}} = \underset{O}{\boldsymbol{m}} - \underset{O}{\boldsymbol{d}_A} \times \boldsymbol{v}, \tag{2}$$

where $\underset{O}{\boldsymbol{d}_A} = \underset{O}{\boldsymbol{d}} - \underset{A}{\boldsymbol{d}}$ is the relative position of point A from pole O. Let us now introduce the *primal part magnitude* $v = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}$ and what we call the *dual part magnitude* (as it will be pointed out below),

$$m = v^{-1} \boldsymbol{v} \cdot \boldsymbol{c}. \tag{3}$$

Both are pole-independent scalars. Then, using the identities $\boldsymbol{v} \times \boldsymbol{v} \times = \boldsymbol{v} \otimes \boldsymbol{v} - v^2 \boldsymbol{I}$ and $\boldsymbol{v} \times^3 = -v^{-2} \boldsymbol{v} \times$, operate a projection of moment $\underset{A}{\boldsymbol{m}}$, Eq. (2), along vector $\boldsymbol{v}$ and on the plane normal to $\boldsymbol{v}$, $\underset{A}{\boldsymbol{m}} = v^{-2}(\boldsymbol{v} \otimes \boldsymbol{v} - \boldsymbol{v} \times \boldsymbol{v} \times) \cdot (\underset{O}{\boldsymbol{m}} - \underset{O}{\boldsymbol{d}_A} \times \boldsymbol{v}) = (v^{-2} \boldsymbol{v} \cdot \underset{O}{\boldsymbol{m}}) \boldsymbol{v} - v^{-2}(\boldsymbol{v} \times^3 \underset{O}{\boldsymbol{d}_A} + \boldsymbol{v} \times \boldsymbol{v} \times \underset{O}{\boldsymbol{m}})$, and obtain

$$\underset{A}{\boldsymbol{m}} = m v^{-1} \boldsymbol{v} + \boldsymbol{v} \times (\underset{O}{\boldsymbol{d}_A} - v^{-2} \boldsymbol{v} \times \underset{O}{\boldsymbol{m}}).$$

Now, choose a pole A such that

$$\underset{O}{\boldsymbol{d}_A} = p v^{-1} \boldsymbol{v} + v^{-2} \boldsymbol{v} \times \underset{O}{\boldsymbol{m}} \quad (\forall \text{ real length } p). \tag{4}$$

This makes the moment $\underset{A}{\boldsymbol{m}}$ equal to $m v^{-1} \boldsymbol{v}$, a vector parallel to $\boldsymbol{v}$, and allows writing Eq. (2) as

$$\underset{O}{\boldsymbol{m}} = m v^{-1} \boldsymbol{v} + \underset{O}{\boldsymbol{d}_A} \times \boldsymbol{v}, \tag{5}$$

with $\underset{O}{\boldsymbol{d}_A}$ given by Eq. (4) and m by Eq. (3).

According to Eq. (5), the vector-and-couple pair $(\boldsymbol{v}, \boldsymbol{c})$ is *equipollent* to a pair made of a vector $\boldsymbol{v}$ applied at any point of the straight line $\underset{O}{\boldsymbol{d}_A}(p)$ and a *co-axial* couple $m v^{-1} \boldsymbol{v}$ (independent of any pole)—namely a *co-axial applied vector-and-couple* pair $(\boldsymbol{v}, m v^{-1} \boldsymbol{v})$. The oriented straight line $\underset{O}{\boldsymbol{d}_A}(p)$ defined by Eq. (4) is an invariant geometric property of the applied vector-and-couple pair, referred to as the *axis*. In the case $\boldsymbol{c} = \boldsymbol{0}$, the pair reduces to a simple applied vector and the axis coincides with its axode. Instead, in the case $\boldsymbol{v} = \boldsymbol{0}$, the pair reduces to a pure couple and the axis becomes undefined (in fact any direction parallel to $\boldsymbol{c}$ may be assumed as the axis). The decomposition of the rigid motion according to Mozzi–Chasles' theorem (see, for instance, Murray et al., 1994; Borri et al., 2000) is a kinematical example of reduction of an applied vector-and-couple pair to a co-axial applied vector-and-couple pair.

An important corollary follows from the foregoing theorem. *Any proper dual vector can be decomposed into an invariant, pole-independent dual magnitude multiplied by a unit, pole-based dual vector*. In fact, given the multiplicative decomposition $\boldsymbol{v} = v\boldsymbol{e}$ of a vector $\boldsymbol{v}$ and the equipollence statement Eq. (5), it is seen that a dual vector $\boldsymbol{v} + \varepsilon \underset{O}{\boldsymbol{m}}$ can always be written as $v\boldsymbol{e} + \varepsilon(m\boldsymbol{e} + v\underset{O}{\boldsymbol{d}_A} \times \boldsymbol{e}) = (v + \varepsilon m)(\boldsymbol{I} + \varepsilon \underset{O}{\boldsymbol{d}_A}\times)\boldsymbol{e}$. Therefore, any proper dual vector $\underset{O}{\boldsymbol{w}} = \boldsymbol{v} + \varepsilon \underset{O}{\boldsymbol{m}}$ allows a *multiplicative decomposition*

$$\underset{O}{\boldsymbol{w}} = w\underset{O}{\boldsymbol{r}} \tag{6}$$

into an *invariant dual magnitude*

$$w = v + \varepsilon m, \tag{7}$$

where v is the magnitude of $\boldsymbol{v}$ and m the dual part magnitude given by Eq. (3), multiplied by a *unit dual vector*

$$\underset{O}{\boldsymbol{r}} = \underset{O}{\boldsymbol{D}_A}\boldsymbol{e} = (\boldsymbol{I} + \varepsilon \underset{O}{\boldsymbol{d}_A}\times)\boldsymbol{e}, \tag{8}$$

where: $\underset{O}{\boldsymbol{D}_A}$ is the arm of the axis of the applied pair from pole O, the distance $\underset{O}{\boldsymbol{d}_A}$ is given by Eq. (4), and $\boldsymbol{e}$ is the unit vector of $\boldsymbol{v}$. It is worth noting that the inner product of the unit dual vector by itself yields the real *unity*, $\underset{O}{\boldsymbol{r}} \cdot \underset{O}{\boldsymbol{r}} = 1$. As a consequence, $\underset{O}{\boldsymbol{w}} \cdot \underset{O}{\boldsymbol{r}} = w$ and $\underset{O}{\boldsymbol{w}} \cdot \underset{O}{\boldsymbol{w}} = w^2 = v^2 + \varepsilon 2vm$. The multiplicative decomposition is helpful in extending almost all the properties of vectors and tensors to the relevant dual objects. As an exercise, consider extending to the dual case the well-known recurrence formulae of the powers of skew-symmetric tensors (Argyris, 1982; Bottasso and Borri, 1998):

$$\begin{aligned} \underset{O}{\boldsymbol{w}}\times^{2n-1} &= (-1)^{n-1} w^{2(n-1)} \underset{O}{\boldsymbol{w}}\times, \\ \underset{O}{\boldsymbol{w}}\times^{2n} &= (-1)^{n-1} w^{2(n-1)} \underset{O}{\boldsymbol{w}}\times^2 \quad (\forall n \geqslant 0). \end{aligned} \tag{9}$$

2.4. Determinant of dual tensors

Like real tensors, second-order dual tensors possess three dual scalar invariants, namely the coefficients of the eigenvalue equation. They are referred to as the trace (linear), the second invariant (quadratic) and the determinant (cubic), and the relevant explicit expressions are found in Box 2. For a generic pole-based dual tensor $\underset{O}{\boldsymbol{W}} = \boldsymbol{V} + \varepsilon \underset{O}{\boldsymbol{M}}$ with $\underset{O}{\boldsymbol{M}} = \boldsymbol{C} + \underset{O}{\boldsymbol{d}} \times \boldsymbol{V}$, they are in general pole-dependent dual numbers. However, it is noted that this is not true for the *determinant*, $W = \det \underset{O}{\boldsymbol{W}}$, which is non-null when the primal part $\boldsymbol{V}$ is non-singular and results in a *pole-independent scalar invariant*,

$$W = \det \boldsymbol{V}(1 + \varepsilon\,\mathrm{tr}(\boldsymbol{V}^{-1}\boldsymbol{C})), \tag{10}$$

since the term $\mathrm{tr}(\boldsymbol{V}^{-1}\underset{O}{\boldsymbol{d}} \times \boldsymbol{V}) = \mathrm{tr}(\underset{O}{\boldsymbol{d}} \times \boldsymbol{V}\boldsymbol{V}^{-1}) = \mathrm{tr}(\underset{O}{\boldsymbol{d}}\times)$ is obviously null.

2.5. Rototranslation

A rototranslation is defined as *a dual tensor that transforms a pole-based dual vector into another dual vector of the same magnitude based on the same pole*. A linear transform of a dual vector like $\underset{O}{\boldsymbol{w}'} = \underset{O}{\boldsymbol{H}}\underset{O}{\boldsymbol{w}}$ preserves the magnitude, $\underset{O}{\boldsymbol{w}'} \cdot \underset{O}{\boldsymbol{w}'} = \underset{O}{\boldsymbol{w}} \cdot \underset{O}{\boldsymbol{w}}$, if and only if tensor $\underset{O}{\boldsymbol{H}}$ is orthogonal ($\underset{O}{\boldsymbol{H}}\underset{O}{\boldsymbol{H}}^{\mathrm{T}} = \underset{O}{\boldsymbol{H}}^{\mathrm{T}}\underset{O}{\boldsymbol{H}} = \boldsymbol{I}$) and unimodular ($\det \underset{O}{\boldsymbol{H}} = 1$), i.e. *orthonormal* (Stramigioli et al., 2002). To be orthonormal, a dual tensor is required to possess a form like (Angeles, 1998)

$$\underset{O}{\boldsymbol{H}} = \boldsymbol{\Phi} + \varepsilon \underset{O}{\boldsymbol{t}} \times \boldsymbol{\Phi}, \tag{11}$$

with $\boldsymbol{\Phi}$ an orthonormal real tensor, namely a *rotation tensor*. In Eq. (11), $\underset{O}{\boldsymbol{t}}$ is a vector with physical dimension of a length, which has been called *translation vector* (Borri et al., 2000). By introducing the *dual translation tensor*

$$\underset{O}{\boldsymbol{T}} = \boldsymbol{I} + \varepsilon\, \underset{O}{\boldsymbol{t}}\times, \tag{12}$$

again an orthonormal dual tensor, a rototranslation can be written

$$\underset{O}{\boldsymbol{H}} = \underset{O}{\boldsymbol{T}}\boldsymbol{\Phi}. \tag{13}$$

Eq. (13) corresponds to a *multiplicative decomposition* into a rotation $\boldsymbol{\Phi}$ around pole O followed by a translation $\underset{O}{\boldsymbol{T}}$. (Incidentally, the same rototranslation also allows the alternative form $\boldsymbol{\Phi} + \varepsilon\boldsymbol{\Phi}\underset{O}{\boldsymbol{t}}^{*}\times$ leading to the decomposition $\boldsymbol{\Phi}\underset{O}{\boldsymbol{T}}^{*}$ involving a translation tensor $\underset{O}{\boldsymbol{T}}^{*}$ and vector $\underset{O}{\boldsymbol{t}}^{*}$ such that $\underset{O}{\boldsymbol{t}} = \boldsymbol{\Phi}\underset{O}{\boldsymbol{t}}^{*}$.) Applying a rototranslation to a dual tensor transforms its dyadic component dual vectors by preserving their magnitude, for instance $\underset{O}{\boldsymbol{W}}' = \underset{O}{\boldsymbol{H}}\underset{O}{\boldsymbol{W}} = \underset{O}{\boldsymbol{H}}\underset{O}{\boldsymbol{W}}_j \otimes \boldsymbol{g}^j = \underset{O}{\boldsymbol{W}}'_j \otimes \boldsymbol{g}^j$, hence it transforms a dual tensor by preserving its determinant.

It is worth stressing that a rototranslation is the composition of two subsequent particular rototranslations, the rotation around the pole (a pure real rototranslation) and the dual translation, resulting in general in a *pole-dependent* transform. By changing the pole, the same transform will be represented by a different dual tensor, which however is still a rototranslation, owing to the orthogonality property of the arm operator. With respect to a new pole Q, for instance, the same rototranslation would transform as $\underset{Q}{\boldsymbol{H}} = \underset{Q}{\boldsymbol{D}_O}\underset{O}{\boldsymbol{H}}\underset{Q}{\boldsymbol{D}}{}_O^{\mathrm{T}}$, resulting in $\underset{Q}{\boldsymbol{H}} = \underset{Q}{\boldsymbol{T}}\boldsymbol{\Phi}$ with $\underset{Q}{\boldsymbol{T}} = \boldsymbol{I} + \varepsilon\,\underset{Q}{\boldsymbol{t}}\times$ and $\underset{Q}{\boldsymbol{t}} = \underset{O}{\boldsymbol{t}} + (\boldsymbol{\Phi} - \boldsymbol{I})\underset{O}{\boldsymbol{d}}{}_Q$, where $\underset{O}{\boldsymbol{d}}{}_Q = \underset{O}{\boldsymbol{d}} - \underset{Q}{\boldsymbol{d}}$ is the relative position of pole Q from pole O. From the relation $\underset{Q}{\boldsymbol{t}} = \underset{O}{\boldsymbol{t}} + (\boldsymbol{\Phi} - \boldsymbol{I})\underset{O}{\boldsymbol{d}}{}_Q$ between differently based translation vectors, it is clear that a rototranslation $\underset{O}{\boldsymbol{T}}\boldsymbol{\Phi}$ such that $\underset{O}{\boldsymbol{t}} = -(\boldsymbol{\Phi} - \boldsymbol{I})\underset{O}{\boldsymbol{d}}{}_Q$ is a pure rotation $\boldsymbol{\Phi}$ around a point Q. Therefore, three particular cases of rototranslation are outlined (see also Eqs. (21)):

- the pure rotation around the application point: $\underset{O}{\boldsymbol{H}} = \underset{O}{\boldsymbol{T}}\boldsymbol{\Phi}$ with $\underset{O}{\boldsymbol{t}} = -(\boldsymbol{\Phi} - \boldsymbol{I})\underset{O}{\boldsymbol{d}} = -(\boldsymbol{\Phi} - \boldsymbol{I})\underset{O}{\boldsymbol{x}}$;
- the pure rotation around the pole: $\underset{O}{\boldsymbol{H}} = \boldsymbol{\Phi}$ and $\underset{O}{\boldsymbol{T}} = \boldsymbol{I}$ (i.e. $\underset{O}{\boldsymbol{t}} = \boldsymbol{0}$);
- the pure translation: $\underset{O}{\boldsymbol{H}} = \underset{O}{\boldsymbol{T}}$ with $\underset{O}{\boldsymbol{t}} = \boldsymbol{u} = \underset{O}{\boldsymbol{x}}' - \underset{O}{\boldsymbol{x}}$ (a pole-independent displacement) and $\boldsymbol{\Phi} = \boldsymbol{I}$.

Rototranslations behave exactly like rotations. They compose multiplicatively and do not commute. Moreover, they allow the form of the exponential of a skew-symmetric dual tensor built on a dual vector, leading to the concept of the *exponential map* of rototranslation and the associated *differential map*, and then to the relevant natural parameterization. These topics have been discussed in detail by Borri and coworkers in the field of multibody dynamics and by several authors (for instance Murray et al., 1994; Angeles, 1998) in the field of robotics. The reader is referred to the recent papers by Borri and Bottasso (1998), Bottasso and Borri (1998), Borri et al. (2000), Bauchau and Trainelli (2003) and Trainelli (2002) for a deep and exhaustive exposition of the matter for both rotations and rototranslations. Here, we just present the main issues written in terms of dual entities.

First, remember the exponential map of the rotation and the associated differential, or tangent, map. They can be stated as

$$\begin{aligned} \boldsymbol{\Phi} &= \exp(\boldsymbol{\varphi}\times) = \sum_{k=0}^{\infty} \frac{1}{k!}\boldsymbol{\varphi}\times^{k}, \\ \boldsymbol{\Gamma} &= \operatorname{dexp}(\boldsymbol{\varphi}\times) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!}\boldsymbol{\varphi}\times^{k}, \end{aligned} \tag{14}$$

where $\boldsymbol{\Phi}$ is a rotation tensor, $\boldsymbol{\varphi} = \varphi\boldsymbol{k}$ is the relevant *rotation vector* (with the rotation angle φ as magnitude and $\boldsymbol{k}$ as unit vector) and $\boldsymbol{\Gamma}$ is the relevant differential map. With reference to the modern differential geometry, the exponential map is the function relating the special orthogonal Lie group to its Lie algebra, namely the rotation tensor $\boldsymbol{\Phi} \in \mathrm{SO}(3)$ to the skew-symmetric tensor $\boldsymbol{\varphi}\times \in \mathrm{so}(3)$. It is a surjective map and its multi-valued inverse is the logarithmic map, $\boldsymbol{\varphi}\times = \log\boldsymbol{\Phi}$, a function enabling to extract the rotation vector $\boldsymbol{\varphi}$ as an axial vector, namely $\boldsymbol{\varphi} = \mathrm{ax}\log\boldsymbol{\Phi}$. The logarithmic map also provides a direct relation between the differential and exponential maps, $\boldsymbol{\Gamma} = \mathrm{dexp}\log\boldsymbol{\Phi}$. The differential map is the function relating the derivative of the group element with the derivative of the algebra element (Borri et al., 2000), as shown by the well-known differentiation formula $\mathrm{d}\boldsymbol{\Phi} = (\boldsymbol{\Gamma}\mathrm{d}\boldsymbol{\varphi}) \times \boldsymbol{\Phi} = (\boldsymbol{\Gamma}\,\mathrm{ax}\,\mathrm{d}(\log\boldsymbol{\Phi})) \times \boldsymbol{\Phi}$ (e.g. Ibrahimbegović et al., 1995; Bottasso and Borri, 1998). In our work we will exploit plain tensor algebra and avoid using the abstract formalism of the group theory. By the way, it can be useful to note that the exponential and differential maps are the evolution and convolution operators, respectively, of the solution of an initial value problem governed by an ordinary differential equation with constant coefficients (Borri et al., 2000).

The exponential map of the rototranslation and the associated differential map are stated as follows (recall Box 2),

$$\begin{aligned} \underset{O}{\boldsymbol{H}} &= \exp(\underset{O}{\boldsymbol{\eta}}\times) = \sum_{k=0}^{\infty} \frac{1}{k!} \underset{O}{\boldsymbol{\eta}}\times^{k} \\ &= \exp(\boldsymbol{\varphi}\times + \varepsilon \underset{O}{\boldsymbol{\rho}}\times) = \sum_{k=0}^{\infty} \frac{1}{k!} \boldsymbol{\varphi}\times^{k} + \varepsilon \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{1}{(n+m+1)!} \boldsymbol{\varphi}\times^{n} \underset{O}{\boldsymbol{\rho}}\times \boldsymbol{\varphi}\times^{m}, \\ \underset{O}{\boldsymbol{\Lambda}} &= \mathrm{dexp}(\underset{O}{\boldsymbol{\eta}}\times) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \underset{O}{\boldsymbol{\eta}}\times^{k} \\ &= \mathrm{dexp}(\boldsymbol{\varphi}\times + \varepsilon \underset{O}{\boldsymbol{\rho}}\times) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \boldsymbol{\varphi}\times^{k} + \varepsilon \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{1}{(n+m+2)!} \boldsymbol{\varphi}\times^{n} \underset{O}{\boldsymbol{\rho}}\times \boldsymbol{\varphi}\times^{m}, \end{aligned} \tag{15}$$

and render dual tensors by taking as argument a skew-symmetric dual tensor built on dual vector

$$\underset{O}{\boldsymbol{\eta}} = \boldsymbol{\varphi} + \varepsilon \underset{O}{\boldsymbol{\rho}}. \tag{16}$$

The extension of the Lie group theory to such geometrical objects is straightforward, see for instance Murray et al. (1994), Borri et al. (2000), Stramigioli et al. (2002).

Dual vector $\underset{O}{\boldsymbol{\eta}}$ is made of an *angular* part $\boldsymbol{\varphi}$ and a pole-dependent *linear* part $\underset{O}{\boldsymbol{\rho}}$, and will be referred to as the *helix* of the rototranslation $\underset{O}{\boldsymbol{H}}$. The primal part is just the rotation vector of the rotation $\boldsymbol{\Phi}$, while the dual part is, in general, the sum $\underset{O}{\boldsymbol{\rho}} = \boldsymbol{u} + \underset{O}{\boldsymbol{d}} \times \boldsymbol{\varphi}$ of a displacement $\boldsymbol{u}$ and the 'moment' of $\boldsymbol{\varphi}$ with respect to pole O. The pair $(\boldsymbol{\varphi}, \boldsymbol{u})$ is equipollent to a rotation $\boldsymbol{\varphi}$ around the *axis* of the helix $\underset{O}{\boldsymbol{d}_A}(p) = p\varphi^{-1}\boldsymbol{\varphi} + \varphi^{-2}\boldsymbol{\varphi} \times \underset{O}{\boldsymbol{\rho}}$ ($\forall$ real p), and a *co-axial* displacement $\rho\varphi^{-1}\boldsymbol{\varphi}$, with $\rho = \varphi^{-1}\boldsymbol{\varphi}\cdot\boldsymbol{u}$ the *linear* magnitude (the Mozzi–Chasles' theorem). The *magnitude* of the helix $\underset{O}{\boldsymbol{\eta}} = \eta\underset{O}{\boldsymbol{h}}$ is the pole-independent dual scalar $\eta = \varphi + \varepsilon\rho$, and the *helix unit dual vector* $\underset{O}{\boldsymbol{h}} = \underset{O}{\boldsymbol{D}_A}\boldsymbol{k} = (\boldsymbol{I} + \varepsilon\underset{O}{\boldsymbol{d}_A}\times)\boldsymbol{k}$ is the transport of the rotation unit vector $\boldsymbol{k}$ from the helix axis to pole O. Another expression for the helix magnitude is $\eta = (1 + \varepsilon\tau)\varphi$, involving the helix pitch $\tau = \varphi^{-1}\rho = \varphi^{-2}\boldsymbol{\varphi}\cdot\boldsymbol{u}$.

Of course, the exponential map applies separately to both factors of the multiplicative decomposition Eq. (13). In fact, from Eqs. (12) and (14) and due to the property $\varepsilon^k = 0$ for $k > 1$, it is easily seen that $\underset{O}{\boldsymbol{T}} = \exp(\varepsilon\underset{O}{\boldsymbol{t}}\times)$ and $\mathrm{dexp}(\varepsilon\underset{O}{\boldsymbol{t}}\times) = \boldsymbol{I} + \varepsilon\frac{1}{2}\underset{O}{\boldsymbol{t}}\times$. Therefore, with reference to Eqs. (11) and (13), other expressions for the rototranslation involving vectors $\boldsymbol{\varphi}$ and $\underset{O}{\boldsymbol{t}}$ are obtained,

$$\underset{O}{\boldsymbol{H}} = \exp(\varepsilon\, \underset{O}{\boldsymbol{t}}\times)\exp(\boldsymbol{\varphi}\times) = \exp(\boldsymbol{\varphi}\times) + \varepsilon\, \underset{O}{\boldsymbol{t}} \times \exp(\boldsymbol{\varphi}\times). \tag{17}$$

So, either the translation vector $\underset{O}{\boldsymbol{t}}$ or the linear part $\underset{O}{\boldsymbol{\rho}}$ of the helix arise as significant 'moments' for the vectorial representation of the rototranslation. They are found to be related to each other by the same mapping tensor $\boldsymbol{\Gamma}$ of the differential map of the rotation,

$$\underset{O}{\boldsymbol{t}} = \boldsymbol{\Gamma}\underset{O}{\boldsymbol{\rho}}, \tag{18}$$

as it will be shown in Part II, Appendix B. In next section, such alternative measures of displacement will be related to the standard displacement vector.

2.6. Frames

In this paper, dual vectors and tensors are widely used to represent many physical quantities belonging to either the kinematic space or the dual space of the acting forces. One basic quantity for the description of the continuum, however, is rather geometrical in character, so it is briefly discussed in this section.

Three differently oriented, non-null vectors located at a point in the space constitute an applied frame. Their dyadic composition with any appropriate base gives a non-singular tensor. A non-singular tensor $\boldsymbol{G}$ applied at a point of relative position $\underset{O}{\boldsymbol{x}}$ from a pole O is fully represented by the dual tensor

$$\underset{O}{\boldsymbol{F}} = \underset{O}{\boldsymbol{X}}\boldsymbol{G} = (\boldsymbol{I} + \varepsilon\underset{O}{\boldsymbol{x}}\times)\boldsymbol{G}, \tag{19}$$

which can be referred to as a *(dual) frame*. $\underset{O}{\boldsymbol{F}}$ is a non-singular dual tensor lacking of the pure couple part. The orthonormal dual tensor $\underset{O}{\boldsymbol{X}}$, that can be called the *position tensor*, acts as the transport of tensor $\boldsymbol{G}$ to the pole, so that the dual frame $\underset{O}{\boldsymbol{F}}$ turns out to be the pole-based version of the self-based frame $\boldsymbol{G}$.

It is worth noting that the determinant of the dual frame $\underset{O}{\boldsymbol{F}}$ coincides with the determinant of tensor $\boldsymbol{G}$, say $g = \det\boldsymbol{G}$, see Eq. (10). Moreover, given a dyadic representation like $\boldsymbol{G} = \boldsymbol{g}_j \otimes \boldsymbol{i}^j$ and correspondingly $\underset{O}{\boldsymbol{F}} = \underset{O}{\boldsymbol{f}}_j \otimes \boldsymbol{i}^j = \underset{O}{\boldsymbol{X}}\boldsymbol{g}_j \otimes \boldsymbol{i}^j$, it is seen that the same determinant can be also computed as the frame volume $g = \boldsymbol{g}_1 \times \boldsymbol{g}_2 \cdot \boldsymbol{g}_3 = \underset{O}{\boldsymbol{f}}_1 \times \underset{O}{\boldsymbol{f}}_2 \cdot \underset{O}{\boldsymbol{f}}_3$. The frame determinant allows computing the determinant of a dual tensor $\underset{O}{\boldsymbol{W}}$, Eq. (10), by means of the usual rule as $W = \det\underset{O}{\boldsymbol{W}} = g^{-1}(\underset{O}{\boldsymbol{W}}\underset{O}{\boldsymbol{f}}_1) \times (\underset{O}{\boldsymbol{W}}\underset{O}{\boldsymbol{f}}_2) \cdot (\underset{O}{\boldsymbol{W}}\underset{O}{\boldsymbol{f}}_3)$. As an application, consider the extension to the dual case of a well-known property of the cross product,

$$(\underset{O}{\boldsymbol{W}}\underset{O}{\boldsymbol{a}})\times = W\underset{O}{\boldsymbol{W}}^{-\mathrm{T}}\underset{O}{\boldsymbol{a}} \times \underset{O}{\boldsymbol{W}}^{-1} \quad (\forall \underset{O}{\boldsymbol{a}}). \tag{20}$$

Rototranslating a dual frame yields another dual frame made of the same tensor, rotated and applied at another point, for instance $\underset{O}{\boldsymbol{F}}' = \underset{O}{\boldsymbol{H}}\underset{O}{\boldsymbol{F}} = \underset{O}{\boldsymbol{T}}\boldsymbol{\Phi}\underset{O}{\boldsymbol{X}}\boldsymbol{G} = \underset{O}{\boldsymbol{X}}'\boldsymbol{G}'$, with $\underset{O}{\boldsymbol{X}}' = \boldsymbol{I} + \varepsilon\underset{O}{\boldsymbol{x}}'\times$ and $\boldsymbol{G}' = \boldsymbol{\Phi}\boldsymbol{G}$. The frame determinant is preserved. As a consequence, the rototranslation between two frames, and the relevant translation vector, can always be given the forms

$$\begin{aligned} \underset{O}{\boldsymbol{H}} &= \underset{O}{\boldsymbol{X}}'\boldsymbol{\Phi}\underset{O}{\boldsymbol{X}}^{\mathrm{T}}, \\ \underset{O}{\boldsymbol{t}} &= \underset{O}{\boldsymbol{x}}' - \boldsymbol{\Phi}\underset{O}{\boldsymbol{x}} \end{aligned} \tag{21}$$

with $\boldsymbol{\Phi} = \boldsymbol{G}'\boldsymbol{G}^{-1}$. Notice that, if the frame tensor $\boldsymbol{G}$ is orthonormal, the dual frame itself corresponds to a rototranslation from the identity tensor applied at the pole.

It can be seen from Eq. $(21)_2$ that the translation vector $\underset{O}{\boldsymbol{t}}$ is far different from the *displacement* vector $\boldsymbol{u} = \underset{O}{\boldsymbol{x}}' - \underset{O}{\boldsymbol{x}}$, which is the customary pole-independent measure of displacement in continuum mechanics. The pole-based translation vector $\underset{O}{\boldsymbol{t}}$, and so the linear part $\underset{O}{\boldsymbol{\rho}}$ of the helix, however, are endowed with an important property that is greatly appreciated in multibody dynamics: they are a global measure of a rigid displacement, for they are unique for all the particles of a moving rigid body (Borri et al., 2000). This property strengthens the choice of the term translation for vector $\underset{O}{\boldsymbol{t}}$. The relations between the displacement $\boldsymbol{u}$ and the pole-based measures follow from Eqs. $(21)_2$ and (18) and from the identity $\boldsymbol{\Phi} = \boldsymbol{I} + \boldsymbol{\Gamma}\boldsymbol{\varphi}\times$:

$$\begin{aligned} \boldsymbol{u} &= \underset{O}{\boldsymbol{t}} + (\boldsymbol{\Phi} - \boldsymbol{I})\underset{O}{\boldsymbol{x}} \\ &= \boldsymbol{\Gamma}(\underset{O}{\boldsymbol{\rho}} - \underset{O}{\boldsymbol{x}} \times \boldsymbol{\varphi}). \end{aligned} \tag{22}$$

2.7. Notation

In the remainder of the paper, dual vectors and tensors are always based on a unique pole. Occasionally, the relevant self-based counterparts are defined, however with different symbols. Therefore, the explicit reference to the pole becomes superfluous and from now on we will avoid using under-posed letters.

3. Differentiation of rotation and rototranslation

In the forthcoming variational formulation, we are involved with three mixed but independent differentiations of the rototranslation, namely a virtual differentiation inherent in the variational setting, an incremental differentiation relevant to the solution procedure, and a spatial differentiation describing the curvatures. Therefore, we must be able to differentiate rotations and rototranslations up to third-order. By recursive differentiations of the orthogonality condition, the differential vectors characterizing each subsequent variation of an orthonormal tensor can be identified. They are introduced as characteristic vectors intrinsic to an orthonormal tensor, and are independent of any particular parameterization.

3.1. Rotations

Let us start with the case of rotations. We take a single variation δ, a double variation $\partial\delta$, and finally a triple variation $d\partial\delta$ of the orthogonality condition $\boldsymbol{\Phi}\boldsymbol{\Phi}^T = \boldsymbol{I}$, and look in sequence at tensors $\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^T$, $\partial\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^T$ and $d\partial\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^T$; see Merlini (2002) for details. After the first variation, $\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^T$ is seen to be a skew-symmetric tensor, so $\delta\boldsymbol{\Phi}$ is recognized to be determined by the relevant axial vector. This is given the symbol $\boldsymbol{\varphi}_\delta$ and is actually an infinitesimal, differential vector although it is by no means the differential of a rotation vector. This is a well-known issue, see for instance Borri and Bottasso (1994a) and Ibrahimbegović et al. (1995). After the second variation, the symmetric part of tensor $\partial\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^T$ is seen to depend on the (first) differential vectors $\boldsymbol{\varphi}_\partial$ and $\boldsymbol{\varphi}_\delta$, while the skew-symmetric part depends also on their variations $\partial\boldsymbol{\varphi}_\delta$ and $\delta\boldsymbol{\varphi}_\partial$. So, the axial vector of tensor $\partial\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^T$ can be taken as a further characteristic differential vector, referred to as the second differential vector, $\boldsymbol{\varphi}_{\partial\delta}$, relevant to the second variation $\partial\delta\boldsymbol{\Phi}$. And so on, the symmetric part of tensor $d\partial\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^T$ is seen to depend again just on the lower-order differential vectors $\boldsymbol{\varphi}_d$, $\boldsymbol{\varphi}_\partial$, $\boldsymbol{\varphi}_\delta$, $\boldsymbol{\varphi}_{d\partial}$, $\boldsymbol{\varphi}_{\partial\delta}$ and $\boldsymbol{\varphi}_{\delta d}$, while the skew-symmetric part depends also on their first and second variations. Again, the axial vector of tensor $d\partial\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^T$ can be assumed as the third characteristic differential vector, $\boldsymbol{\varphi}_{d\partial\delta}$, relevant to the third variation $d\partial\delta\boldsymbol{\Phi}$. Since this mechanism applies to any differentiation order, it seems appropriate to reserve the role of characteristic differential vectors to the axial vectors mentioned above, i.e.

$$\boldsymbol{\varphi}_\delta = \mathrm{ax}(\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^T),$$

$$\boldsymbol{\varphi}_{\partial\delta} = \mathrm{ax}(\partial\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^T),$$

$$\boldsymbol{\varphi}_{\mathrm{d}\partial\delta} = \mathrm{ax}(\mathrm{d}\partial\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}).$$

This way, the subsequent variations of the rotation (up to third-order) become characterized recursively as follows:

$$\begin{aligned}
&\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}} = \boldsymbol{\varphi}_{\delta}\times,\\
&\partial\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}} = \boldsymbol{\varphi}_{\partial\delta}\times + \tfrac{1}{2}(\boldsymbol{\varphi}_{\partial}\times\boldsymbol{\varphi}_{\delta}\times + \boldsymbol{\varphi}_{\delta}\times\boldsymbol{\varphi}_{\partial}\times),\\
&\mathrm{d}\partial\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}} = \boldsymbol{\varphi}_{\mathrm{d}\partial\delta}\times + \tfrac{1}{2}(\boldsymbol{\varphi}_{\mathrm{d}\partial}\times\boldsymbol{\varphi}_{\delta}\times + \boldsymbol{\varphi}_{\partial\delta}\times\boldsymbol{\varphi}_{\mathrm{d}}\times + \boldsymbol{\varphi}_{\delta\mathrm{d}}\times\boldsymbol{\varphi}_{\partial}\times + \boldsymbol{\varphi}_{\mathrm{d}}\times\boldsymbol{\varphi}_{\partial\delta}\times + \boldsymbol{\varphi}_{\partial}\times\boldsymbol{\varphi}_{\delta\mathrm{d}}\times + \boldsymbol{\varphi}_{\delta}\times\boldsymbol{\varphi}_{\mathrm{d}\partial}\times).
\end{aligned} \tag{23}$$

A collection of equivalent expressions for the second and third characteristic differential vectors is given:

$$\boldsymbol{\varphi}_{\partial\delta}\begin{cases} = \partial\boldsymbol{\varphi}_{\delta} - \frac{1}{2}\boldsymbol{\varphi}_{\partial}\times\boldsymbol{\varphi}_{\delta}\\ = \delta\boldsymbol{\varphi}_{\partial} - \frac{1}{2}\boldsymbol{\varphi}_{\delta}\times\boldsymbol{\varphi}_{\partial}\,,\\ = \frac{1}{2}(\partial\boldsymbol{\varphi}_{\delta} + \delta\boldsymbol{\varphi}_{\partial})\end{cases} \tag{24}$$

$$\boldsymbol{\varphi}_{\mathrm{d}\partial\delta}\begin{cases}
= \mathrm{d}\boldsymbol{\varphi}_{\partial\delta} - \frac{1}{2}\boldsymbol{\varphi}_{\mathrm{d}}\times\boldsymbol{\varphi}_{\partial\delta} - \frac{1}{4}(2\boldsymbol{\varphi}_{\mathrm{d}}\otimes\boldsymbol{\varphi}_{\partial}\cdot\boldsymbol{\varphi}_{\delta} + \boldsymbol{\varphi}_{\partial}\otimes\boldsymbol{\varphi}_{\delta}\cdot\boldsymbol{\varphi}_{\mathrm{d}} + \boldsymbol{\varphi}_{\delta}\otimes\boldsymbol{\varphi}_{\mathrm{d}}\cdot\boldsymbol{\varphi}_{\partial})\\
= \partial\boldsymbol{\varphi}_{\delta\mathrm{d}} - \frac{1}{2}\boldsymbol{\varphi}_{\partial}\times\boldsymbol{\varphi}_{\delta\mathrm{d}} - \frac{1}{4}(\boldsymbol{\varphi}_{\mathrm{d}}\otimes\boldsymbol{\varphi}_{\partial}\cdot\boldsymbol{\varphi}_{\delta} + 2\boldsymbol{\varphi}_{\partial}\otimes\boldsymbol{\varphi}_{\delta}\cdot\boldsymbol{\varphi}_{\mathrm{d}} + \boldsymbol{\varphi}_{\delta}\otimes\boldsymbol{\varphi}_{\mathrm{d}}\cdot\boldsymbol{\varphi}_{\partial})\\
= \delta\boldsymbol{\varphi}_{\mathrm{d}\partial} - \frac{1}{2}\boldsymbol{\varphi}_{\delta}\times\boldsymbol{\varphi}_{\mathrm{d}\partial} - \frac{1}{4}(\boldsymbol{\varphi}_{\mathrm{d}}\otimes\boldsymbol{\varphi}_{\partial}\cdot\boldsymbol{\varphi}_{\delta} + \boldsymbol{\varphi}_{\partial}\otimes\boldsymbol{\varphi}_{\delta}\cdot\boldsymbol{\varphi}_{\mathrm{d}} + 2\boldsymbol{\varphi}_{\delta}\otimes\boldsymbol{\varphi}_{\mathrm{d}}\cdot\boldsymbol{\varphi}_{\partial})\\
= \frac{1}{3}(\mathrm{d}\boldsymbol{\varphi}_{\partial\delta} + \partial\boldsymbol{\varphi}_{\delta\mathrm{d}} + \delta\boldsymbol{\varphi}_{\mathrm{d}\partial}) - \frac{1}{6}(\boldsymbol{\varphi}_{\mathrm{d}}\times\boldsymbol{\varphi}_{\partial\delta} + \boldsymbol{\varphi}_{\partial}\times\boldsymbol{\varphi}_{\delta\mathrm{d}} + \boldsymbol{\varphi}_{\delta}\times\boldsymbol{\varphi}_{\mathrm{d}\partial})\\
\quad - \frac{1}{3}(\boldsymbol{\varphi}_{\mathrm{d}}\otimes\boldsymbol{\varphi}_{\partial}\cdot\boldsymbol{\varphi}_{\delta} + \boldsymbol{\varphi}_{\partial}\otimes\boldsymbol{\varphi}_{\delta}\cdot\boldsymbol{\varphi}_{\mathrm{d}} + \boldsymbol{\varphi}_{\delta}\otimes\boldsymbol{\varphi}_{\mathrm{d}}\cdot\boldsymbol{\varphi}_{\partial})\\
= \mathrm{d}\partial\boldsymbol{\varphi}_{\delta} - \frac{1}{2}(\boldsymbol{\varphi}_{\mathrm{d}}\times\boldsymbol{\varphi}_{\partial\delta} + \boldsymbol{\varphi}_{\partial}\times\boldsymbol{\varphi}_{\delta\mathrm{d}} - \boldsymbol{\varphi}_{\delta}\times\boldsymbol{\varphi}_{\mathrm{d}\partial}) - \frac{1}{2}(\boldsymbol{\varphi}_{\mathrm{d}}\otimes\boldsymbol{\varphi}_{\partial}\cdot\boldsymbol{\varphi}_{\delta} + \boldsymbol{\varphi}_{\partial}\otimes\boldsymbol{\varphi}_{\delta}\cdot\boldsymbol{\varphi}_{\mathrm{d}})\\
= \partial\delta\boldsymbol{\varphi}_{\mathrm{d}} - \frac{1}{2}(-\boldsymbol{\varphi}_{\mathrm{d}}\times\boldsymbol{\varphi}_{\partial\delta} + \boldsymbol{\varphi}_{\partial}\times\boldsymbol{\varphi}_{\delta\mathrm{d}} + \boldsymbol{\varphi}_{\delta}\times\boldsymbol{\varphi}_{\mathrm{d}\partial}) - \frac{1}{2}(\boldsymbol{\varphi}_{\partial}\otimes\boldsymbol{\varphi}_{\delta}\cdot\boldsymbol{\varphi}_{\mathrm{d}} + \boldsymbol{\varphi}_{\delta}\otimes\boldsymbol{\varphi}_{\mathrm{d}}\cdot\boldsymbol{\varphi}_{\partial})\\
= \delta\mathrm{d}\boldsymbol{\varphi}_{\partial} - \frac{1}{2}(\boldsymbol{\varphi}_{\mathrm{d}}\times\boldsymbol{\varphi}_{\partial\delta} - \boldsymbol{\varphi}_{\partial}\times\boldsymbol{\varphi}_{\delta\mathrm{d}} + \boldsymbol{\varphi}_{\delta}\times\boldsymbol{\varphi}_{\mathrm{d}\partial}) - \frac{1}{2}(\boldsymbol{\varphi}_{\delta}\otimes\boldsymbol{\varphi}_{\mathrm{d}}\cdot\boldsymbol{\varphi}_{\partial} + \boldsymbol{\varphi}_{\mathrm{d}}\otimes\boldsymbol{\varphi}_{\partial}\cdot\boldsymbol{\varphi}_{\delta})\\
= \frac{1}{3}(\mathrm{d}\partial\boldsymbol{\varphi}_{\delta} + \partial\delta\boldsymbol{\varphi}_{\mathrm{d}} + \delta\mathrm{d}\boldsymbol{\varphi}_{\partial}) - \frac{1}{6}(\boldsymbol{\varphi}_{\mathrm{d}}\times\boldsymbol{\varphi}_{\partial\delta} + \boldsymbol{\varphi}_{\partial}\times\boldsymbol{\varphi}_{\delta\mathrm{d}} + \boldsymbol{\varphi}_{\delta}\times\boldsymbol{\varphi}_{\mathrm{d}\partial})\\
\quad - \frac{1}{3}(\boldsymbol{\varphi}_{\mathrm{d}}\otimes\boldsymbol{\varphi}_{\partial}\cdot\boldsymbol{\varphi}_{\delta} + \boldsymbol{\varphi}_{\partial}\otimes\boldsymbol{\varphi}_{\delta}\cdot\boldsymbol{\varphi}_{\mathrm{d}} + \boldsymbol{\varphi}_{\delta}\otimes\boldsymbol{\varphi}_{\mathrm{d}}\cdot\boldsymbol{\varphi}_{\partial})
\end{cases}. \tag{25}$$

(Also notice the identity $\mathrm{d}\boldsymbol{\varphi}_{\partial\delta} + \partial\boldsymbol{\varphi}_{\delta\mathrm{d}} + \delta\boldsymbol{\varphi}_{\mathrm{d}\partial} = \mathrm{d}\partial\boldsymbol{\varphi}_{\delta} + \partial\delta\boldsymbol{\varphi}_{\mathrm{d}} + \delta\mathrm{d}\boldsymbol{\varphi}_{\partial}$.)

Vector $\boldsymbol{\varphi}_{\delta}$ is often referred to as the virtual rotation vector, so it seems appropriate extending the terminology of *differential rotation vectors* to all the characteristic vectors introduced above. However, it is worth noting that no parameterization of the rotation tensor is implied in the foregoing characterization. Rather, the choice of resorting to the same name and symbol used for the rotation vector is supported by the following reasoning. By taking the relative rotation between a generic rotation $\boldsymbol{\Phi}$ and a varied rotation $\boldsymbol{\Phi} + \delta\boldsymbol{\Phi}$, it ensues $(\boldsymbol{\Phi} + \delta\boldsymbol{\Phi})\boldsymbol{\Phi}^{\mathrm{T}} = \boldsymbol{I} + \boldsymbol{\varphi}_{\delta}\times$, which is a tensor matching exactly the rotation $\exp\boldsymbol{\varphi}_{\delta}\times$ any time vector $\boldsymbol{\varphi}_{\delta}$ is an infinitesimal vector. Of course, a parameterization of the rotation is finally needed in order to evaluate the differential rotation vectors from the variations of the variables chosen as independent variables.

Meaningful relations also hold between the differential rotation vectors and the relevant corotational variations. Corotational variations (which can be interpreted with the formalism of the Lie derivatives) provide objective variation measures and are found in most modern works on geometrically non-linear mechanics, see for instance Borri and Bottasso (1994a,b), Stolarski et al. (1995) or Ibrahimbegović et al. (1995). Some useful formulae for the corotational variations follow,

$$\begin{aligned}
&\boldsymbol{\Phi}\partial(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\delta})\begin{cases} = \delta\boldsymbol{\varphi}_{\partial} \\ = \partial\boldsymbol{\varphi}_{\delta} - \boldsymbol{\varphi}_{\partial}\times\boldsymbol{\varphi}_{\delta} \\ = \boldsymbol{\varphi}_{\partial\delta} - \frac{1}{2}\boldsymbol{\varphi}_{\partial}\times\boldsymbol{\varphi}_{\delta} \end{cases}, \\
&\boldsymbol{\Phi}\mathrm{d}(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\partial\delta}) = \mathrm{d}\boldsymbol{\varphi}_{\partial\delta} - \boldsymbol{\varphi}_{\mathrm{d}}\times\boldsymbol{\varphi}_{\partial\delta}, \\
&\boldsymbol{\Phi}\mathrm{d}\partial(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\delta})\begin{cases} = \mathrm{d}\partial\boldsymbol{\varphi}_{\delta} - (\boldsymbol{\varphi}_{\mathrm{d}}\times\boldsymbol{\varphi}_{\partial\delta} + \boldsymbol{\varphi}_{\partial}\times\boldsymbol{\varphi}_{\delta\mathrm{d}} - \boldsymbol{\varphi}_{\delta}\times\boldsymbol{\varphi}_{\mathrm{d}\partial}) \\ = \boldsymbol{\varphi}_{\mathrm{d}\partial\delta} - \frac{1}{2}(\boldsymbol{\varphi}_{\mathrm{d}}\times\boldsymbol{\varphi}_{\partial\delta} + \boldsymbol{\varphi}_{\partial}\times\boldsymbol{\varphi}_{\delta\mathrm{d}} - \boldsymbol{\varphi}_{\delta}\times\boldsymbol{\varphi}_{\mathrm{d}\partial}) + \frac{1}{2}(\boldsymbol{\varphi}_{\mathrm{d}}\otimes\boldsymbol{\varphi}_{\partial} + \boldsymbol{\varphi}_{\partial}\otimes\boldsymbol{\varphi}_{\mathrm{d}})\cdot\boldsymbol{\varphi}_{\delta} \end{cases}.
\end{aligned} \tag{26}$$

3.2. Rototranslations

When dealing with rototranslations, the same considerations lead to introduce the dual vectors $\boldsymbol{\eta}_{\delta}$, $\boldsymbol{\eta}_{\partial\delta}$ and $\boldsymbol{\eta}_{\mathrm{d}\partial\delta}$ as characteristic differential dual vectors of subsequent orders of differentiation. They are defined as the axial vectors of the following dual tensors,

$$\begin{aligned}
&\delta\boldsymbol{H}\boldsymbol{H}^{\mathrm{T}} = \boldsymbol{\eta}_{\delta}\times, \\
&\partial\delta\boldsymbol{H}\boldsymbol{H}^{\mathrm{T}} = \boldsymbol{\eta}_{\partial\delta}\times + \frac{1}{2}(\boldsymbol{\eta}_{\partial}\times\boldsymbol{\eta}_{\delta}\times + \boldsymbol{\eta}_{\delta}\times\boldsymbol{\eta}_{\partial}\times), \\
&\mathrm{d}\partial\delta\boldsymbol{H}\boldsymbol{H}^{\mathrm{T}} = \boldsymbol{\eta}_{\mathrm{d}\partial\delta}\times + \frac{1}{2}(\boldsymbol{\eta}_{\mathrm{d}\partial}\times\boldsymbol{\eta}_{\delta}\times + \boldsymbol{\eta}_{\partial\delta}\times\boldsymbol{\eta}_{\mathrm{d}}\times + \boldsymbol{\eta}_{\delta\mathrm{d}}\times\boldsymbol{\eta}_{\partial}\times + \boldsymbol{\eta}_{\mathrm{d}}\times\boldsymbol{\eta}_{\partial\delta}\times + \boldsymbol{\eta}_{\partial}\times\boldsymbol{\eta}_{\delta\mathrm{d}}\times + \boldsymbol{\eta}_{\delta}\times\boldsymbol{\eta}_{\mathrm{d}\partial}\times).
\end{aligned} \tag{27}$$

By analogy with the terminology used for the case of rotations, such characteristic dual vectors are called *differential helices*. The same expressions as in Eqs. (24)–(26) apply for them too.

4. Description of the continuum

In this section we develop the fundamentals of the helicoidal modeling. The starting point is a polar description of the continuum as early published by Cosserat and Cosserat (1909) and later developed by several authors, e.g. Toupin (1964) and Kafadar and Eringen (1971). In our investigation, the material particles are assumed as rigid, in the sense that the directors of the oriented medium are not allowed to stretch and rotate relative to each other, as for instance in Ericksen and Truesdell (1958) and Mindlin (1964). So, we are actually concerned with a micropolar medium, which is a special case of the more general micromorphic medium (Eringen and Kafadar, 1976). The polar description we propose departs from the classical scheme (as found in Merlini, 1997) since the configuration of the oriented particle is identified by a single dual tensor that is actually a rototranslation from a reference space frame. The tangent space of such configuration field is obtained by differentiating the rototranslation field, so it turns out to be characterized by a differential helix. This characterization is relevant to both the position of the material particles in a given configuration and the trajectory of one particle along the deformation history, thus the ensuing helicoidal modeling is relevant to both the definition of the material lines within the body and the evolution of the body configurations.

4.1. Particle oriento-position

According to a polar description, a deformable body is a continuum set of infinitesimal yet three-dimensional material particles. The orientation of each particle is identified by means of an embedded triad of independent vectors. In order to unequivocally measure both location and orientation of any particle, an

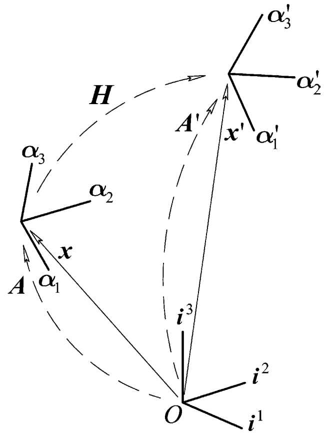

Fig. 2. Identification of position and orientation of a material particle in the reference and current configurations.

absolute reference frame is chosen: as usual and without loss of generality, we refer to the identity tensor $\boldsymbol{I} = \boldsymbol{i}_j \otimes \boldsymbol{i}^j$ made of a triad of orthonormal unit vectors $\boldsymbol{i}_j \equiv \boldsymbol{i}^j$ $(j = 1, 2, 3)$ located at a particular point, namely the origin. So, a particle becomes identified by a position vector $\boldsymbol{x} = x_j\boldsymbol{i}^j$ and an orientation tensor $\boldsymbol{\alpha} = \boldsymbol{\alpha}_j \otimes \boldsymbol{i}^j$ made of the embedded vectors $\boldsymbol{\alpha}_j$ themselves (Fig. 2). The orientation $\boldsymbol{\alpha}$ is a non-singular tensor but does not need to be an orthonormal tensor; however, the independent vectors $\boldsymbol{\alpha}_j$, though arbitrary, are chosen in such a way to form a rigid triad, which is unique for all the particles except for a relative rotation. Thus, the particle orientations can also be understood as rotations from a unique triad. If an orthonormal triad is assumed—in particular, the absolute reference frame in the origin—then the orientation tensors are actually rotation tensors. Although this assumption is not strictly necessary, we shall mostly understand it because it simplifies the formulation without any loss of generality.

A dual frame as defined in Eq. (19) is a single geometrical entity suited to identify as a whole both location and orientation of a particle. This entails the choice of a pole and leads to a pole-based description. Although the choice of the pole is arbitrary, it is natural to assume the origin itself as the pole. So, the position and orientation of a particle become represented by an origin-based dual frame,

$$\boldsymbol{A} = \boldsymbol{X}\boldsymbol{\alpha} = (\boldsymbol{I} + \varepsilon\boldsymbol{x}\times)\boldsymbol{\alpha}, \tag{28}$$

worthy of coining the term *oriento-position*. The primal and dual parts are referred to as the angular part (the orientation $\boldsymbol{\alpha}$) and the linear part of the oriento-position, respectively. Note that, with the assumption of orthonormal triads, the oriento-positions are actually rototranslation tensors. It can also be worth noting that, while the relative positions and orientations between different particles are independent from the absolute reference frame, the relative rototranslations are dependent on the pole and so on the origin—indeed, they do not depend strictly on the origin by itself, which is made coincident with the pole just for convenience.

The particle oriento-position is function of the place, and ultimately of a set of coordinates measuring it. An absolute system of Cartesian orthogonal coordinates is implicitly assumed for computation purposes; it is based on three axes x^j issued from the reference frame in the origin, and on the self-reciprocal frame of constant, orthonormal unit vectors $\boldsymbol{i}_j = \partial\boldsymbol{x}/\partial x^j$. However, an intrinsic system of convective, curvilinear coordinates is explicitly used for the developments; they are measured along three families of smooth material lines ξ^j. Associated with this coordinate system are a frame $\boldsymbol{G} = \boldsymbol{g}_j \otimes \boldsymbol{i}^j$ of covariant base vectors $\boldsymbol{g}_j = \partial\boldsymbol{x}/\partial\xi^j$ and the relevant reciprocal frame $\boldsymbol{G}^{-\mathrm{T}} = \boldsymbol{g}^j \otimes \boldsymbol{i}_j$ of contravariant base vectors $\boldsymbol{g}^j$ (with $\boldsymbol{g}^j \cdot \boldsymbol{g}_k = \delta^j_k$, the Kronecker symbol).

4.2. The helicoidal modeling

The pole-based oriento-position tensor $\boldsymbol{A}$ is an alternative choice of configuration variables to the classical pair, the position vector $\boldsymbol{x}$ and the orientation tensor $\boldsymbol{\alpha}$. However, the non-linearity of the change of variables, Eq. (28), entails remarkable differences as far as the tangent space of the two kinds of fields within the continuum is concerned. By the classical approach, the differential $\mathrm{d}\boldsymbol{x}$ of a linear vector field is considered independently of the differential $\mathrm{d}\boldsymbol{\alpha} = \boldsymbol{\varphi}_\mathrm{d} \times \boldsymbol{\alpha}$ of an orthonormal tensor field (cf. Eq. $(23)_1$ and recall the assumption of an orthonormal orientation field), which is controlled by $\boldsymbol{\varphi}_\mathrm{d}$, a differential rotation vector. By the proposed alternative approach, instead, a unique differential $\mathrm{d}\boldsymbol{A} = \boldsymbol{\eta}_\mathrm{d} \times \boldsymbol{A}$ of an orthonormal dual tensor field (cf. Eq. $(27)_1$), controlled by a differential helix $\boldsymbol{\eta}_\mathrm{d}$, is considered. The explicit form $\boldsymbol{\eta}_\mathrm{d} = \boldsymbol{\varphi}_\mathrm{d} + \varepsilon \boldsymbol{\rho}_\mathrm{d}$ of the differential helix is easily obtained by differentiating the oriento-position tensor defined by Eq. (28). The angular part is seen to be just the differential rotation vector $\boldsymbol{\varphi}_\mathrm{d}$, while the linear part is seen to be the corotational differential of the position vector, $\boldsymbol{\rho}_\mathrm{d} = \boldsymbol{\alpha}\mathrm{d}(\boldsymbol{\alpha}^\mathrm{T}\boldsymbol{x}) = \mathrm{d}\boldsymbol{x} + \boldsymbol{x} \times \boldsymbol{\varphi}_\mathrm{d}$, hence it depends on both the characteristic differentials of the position and orientation fields. That means that the differential of the position is controlled by either parts of the dual differential helix, in fact $\mathrm{d}\boldsymbol{x} = \boldsymbol{\rho}_\mathrm{d} - \boldsymbol{x} \times \boldsymbol{\varphi}_\mathrm{d}$. In other words, the orientation field contributes to define the position itself of near points, and the position and orientation become integrally coupled in a unique field.

Following this approach, we are actually establishing a new modeling of the continuum. It relies on the pole-based oriento-position tensor field as configuration variable, and is referred to as *helicoidal modeling* because a differential helix characterizes the tangent space of the configuration field. A picture of the proposed modeling against the classical one is shown in Fig. 1, where the deep difference between the tangent spaces can be observed; note that either paths have the same, constant angular curvature (see next section).

The explicit dependence on a pole is inherent in this kind of modeling, and the differential helix $\boldsymbol{\eta}_\mathrm{d}$ itself is a pole-based dual vector. It is interesting to analyze the effect of a particular change of pole, from the origin O to the point under consideration, with position vector $\boldsymbol{x}$. The self-based version of $\boldsymbol{\eta}_\mathrm{d}$ is obtainable on use of the arm operator $\boldsymbol{X} = \boldsymbol{I} + \varepsilon \boldsymbol{x}\times$ (recall Eq. (1)) and results in the dual vector $\boldsymbol{X}^\mathrm{T}\boldsymbol{\eta}_\mathrm{d} = \boldsymbol{\varphi}_\mathrm{d} + \varepsilon\, \mathrm{d}\boldsymbol{x}$. The single parts of this dual vector are just the separate characteristic differentials of the orientation and position fields, namely the differentials proper of the classical modeling of the continuum.

4.3. Curvature

Differentiation of the configuration field with respect to the material coordinates leads to define the curvature within the continuum. Once the oriento-position is understood as a rototranslation tensor field, Eq. $(27)_1$ applies as well to each spatial derivative $\partial/\partial\xi^j$ of dual tensor $\boldsymbol{A}$ and can be written $\boldsymbol{A}^\mathrm{T}\boldsymbol{A}_{,j} = (\boldsymbol{A}^\mathrm{T}\boldsymbol{k}_j)\times$. This defines three dual vectors $\boldsymbol{k}_j$ as characterizing the derivatives $\boldsymbol{A}_{,j}$; they can be referred to as the (generalized) curvatures along the coordinate lines. Then, the dyadic composition with the contravariant base vectors $\boldsymbol{g}^j$, namely $\boldsymbol{A}^\mathrm{T}\boldsymbol{A}_{,j} \otimes \boldsymbol{g}^j = (\boldsymbol{A}^\mathrm{T}\boldsymbol{k}_j) \times \otimes\, \boldsymbol{g}^j$, leads to an equation between third-order tensors,

$$\boldsymbol{A}^\mathrm{T}\boldsymbol{A}_{/\otimes} = (\boldsymbol{A}^\mathrm{T}\boldsymbol{k})^\times, \tag{29}$$

that actually defines the *curvature* dual tensor $\boldsymbol{k} = \boldsymbol{k}_j \otimes \boldsymbol{g}^j$ of the continuum. (As usual, we understand the dot symbol of a single contraction in the inner product between tensors).

In Eq. (29), we are introducing two unfamiliar notations. First one is just a concise notation for the spatial derivative operators, i.e. gradient and divergence, inspired by their relation with the outer and inner products between component tensors, respectively (Malvern, 1969): the open derivative $(\ )_{/\otimes} = (\ )_{,j} \otimes \boldsymbol{g}^j$ stands for $\mathrm{grad}(\ ) = (\ ) \otimes \boldsymbol{\nabla}$, and the inner derivative $(\ )_{/\bullet} = (\ )_{,j} \cdot \boldsymbol{g}^j$ will be used later for $\mathrm{div}(\ ) = (\ ) \cdot \boldsymbol{\nabla}$. The second unfamiliar notation is the extension to third-order tensors of the 'vector-cross' notation $(\ )\times$, used to denote a skew-symmetric tensor by means of its axial vector. The 'tensor-cross' notation is defined as $(\ )^\times = (\ )_j \times \otimes \boldsymbol{g}^j$, and $(\ ) = (\ )_j \otimes \boldsymbol{g}^j$ is said the second-order *axial tensor* of third-order tensor $(\ )^\times$. For example, $\boldsymbol{B}^\times = \boldsymbol{B}_j \times \otimes\, \boldsymbol{g}^j$ is the third-order tensor of skew-symmetric nature built with the three second-order skew-symmetric tensors $\boldsymbol{B}_j\times$, hence characterized by the second-order tensor $\boldsymbol{B} = \boldsymbol{B}_j \otimes \boldsymbol{g}^j$ built with the respective axial vectors and referred to as the axial tensor of tensor $\boldsymbol{B}^\times$, namely $\boldsymbol{B} = \mathrm{ax}\,\boldsymbol{B}^\times$. In general, the axial vector of a second-order tensor $\boldsymbol{B}$ characterizes its skew-symmetric part and can be extracted with the formula $\boldsymbol{b} = \mathrm{ax}\boldsymbol{B} = \frac{1}{2}\boldsymbol{g}^j \times \boldsymbol{B}\cdot\boldsymbol{g}_j = \frac{1}{2}\boldsymbol{I}^\times : \boldsymbol{B}$, where $\boldsymbol{I}^\times = \boldsymbol{g}^j \times \otimes \boldsymbol{g}_j$ is the Ricci's tensor. Analogously, the second-order axial tensor of a third-order tensor $\mathscr{B} = \mathscr{B}_j \otimes \boldsymbol{g}^j$ characterizes the part of $\mathscr{B}$ built with the second-order skew-symmetric component tensors $(\mathrm{ax}\mathscr{B}_j)\times$, and can be extracted with the formula $\boldsymbol{B} = \mathrm{ax}\mathscr{B} = \frac{1}{2}\boldsymbol{g}^j \times \mathscr{B} : \boldsymbol{g}_j \otimes \boldsymbol{I} = \frac{1}{2}\boldsymbol{I}^\times : \mathscr{B}$ (Merlini, 1997). Also note that $\boldsymbol{B}_j = (\mathrm{ax}\mathscr{B})_j = \mathrm{ax}\mathscr{B}_j$.

Therefore, the dual tensor $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{k}$ is the axial tensor of the third-order dual tensor of skew-symmetric nature $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}_{/\otimes}$, and the curvature

$$\boldsymbol{k} = \boldsymbol{A}\ \mathrm{ax}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}_{/\otimes}) \tag{30}$$

is the dual tensor characterizing the oriento-position gradient $\boldsymbol{A}_{/\otimes} = \mathrm{d}\boldsymbol{A}/\mathrm{d}\boldsymbol{x}$.

The dual explicit form of the curvature can be obtained from Eqs. (29) and (28):

$$\boldsymbol{k} = \boldsymbol{k}_{\mathrm{a}} + \varepsilon \boldsymbol{k}_{\mathrm{l}} = \boldsymbol{\alpha}\,\mathrm{ax}(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha}_{/\otimes}) + \varepsilon\boldsymbol{\alpha}(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{x})_{/\otimes}. \tag{31}$$

The angular and linear parts are the angular curvature and the corotational gradient of the position vector, respectively. The curvature $\boldsymbol{k}$ is a pole-based dual tensor, and the relevant self-based dual tensor is obtainable using the arm operator $\boldsymbol{X}$,

$$\boldsymbol{X}^{\mathrm{T}}\boldsymbol{k} = \boldsymbol{k}_{\mathrm{a}} + \varepsilon(\boldsymbol{k}_{\mathrm{l}} - \boldsymbol{x}\times\boldsymbol{k}_{\mathrm{a}}) = \boldsymbol{\alpha}\,\mathrm{ax}(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha}_{/\otimes}) + \varepsilon\boldsymbol{x}_{/\otimes}. \tag{32}$$

It is worth noting that the single parts of the self-based version of the curvature, Eq. (32), are respectively the angular curvature $\boldsymbol{k}_{\mathrm{a}}$ characterizing the orientation gradient $\boldsymbol{\alpha}_{/\otimes} = \mathrm{d}\boldsymbol{\alpha}/\mathrm{d}\boldsymbol{x}$, and the position gradient $\boldsymbol{x}_{/\otimes} = \mathrm{d}\boldsymbol{x}/\mathrm{d}\boldsymbol{x}$ (the metric tensor $\boldsymbol{I} = \boldsymbol{g}_j \otimes \boldsymbol{g}^j$). Instead, the linear part of the pole-based curvature $\boldsymbol{k}$ depends on both such tensors, $\boldsymbol{k}_{\mathrm{l}} = \boldsymbol{\alpha}(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{x})_{/\otimes} = \boldsymbol{x}_{/\otimes} + \boldsymbol{x}\times\boldsymbol{k}_{\mathrm{a}}$, and conversely the position gradient is controlled by either parts of $\boldsymbol{k}$, i.e. $\boldsymbol{x}_{/\otimes} = \boldsymbol{k}_{\mathrm{l}} - \boldsymbol{x}\times\boldsymbol{k}_{\mathrm{a}}$. So, the evaluation of neighboring positions according to Eq. (31) is strongly affected by the orientation field, showing again that the position and orientation fields are strictly coupled.

In a sense, the pole-based curvature $\boldsymbol{k}$ from Eq. (31) is the heart of the *helicoidal modeling*. To realize that, consider as an example the simple case of constant curvature $\boldsymbol{k}_j$ along a coordinate line ξ^j and integrate Eq. (29) in that direction from a point with known oriento-position $\boldsymbol{A}_0$, namely solve the problem: $\boldsymbol{A}_{,j} = \boldsymbol{k}_j \times \boldsymbol{A}$, with $\boldsymbol{A} = \boldsymbol{A}_0$ for $\xi^j = 0$. The well-known solution (e.g. Borri et al., 2000) gives $\boldsymbol{A} = \exp(\xi^j\boldsymbol{k}_j\times)\boldsymbol{A}_0$, namely $\boldsymbol{A}$ is a rototranslation of the oriento-position $\boldsymbol{A}_0$ with helix $\xi^j\boldsymbol{k}_j$. In three-dimensional mechanics of deformable solids, a wholly constant curvature tensor $\boldsymbol{k}$ is unlikely; however the same concept can, and will be the base for building consistent finite elements for geometrical non-linearity.

4.4. Deformation

Let us understand that the foregoing geometric description is relevant to the undeformed configuration of the continuum, namely the reference (or initial) configuration. Each different configuration of a deformable body can be described the same way by its own field of oriento-position tensors. In particular, we are considering now a generic, deformed configuration, which is typically an unknown in a problem of continuum mechanics. It will be referred to as the current configuration, and the quantities pertaining to it will be marked by an appended prime ($'$), see Fig. 2.

According to a Lagrangian point of view, we assume as independent variable the initial position and describe the *current oriento-position*

$$\boldsymbol{A}' = \boldsymbol{X}'\boldsymbol{\alpha}' = (\boldsymbol{I} + \varepsilon\boldsymbol{x}'\times)\boldsymbol{\alpha}' \tag{33}$$

as a function of it, namely $\boldsymbol{A}'(\boldsymbol{x})$. Taking the spatial derivatives with respect to the initial coordinates leads to equation

$$\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{A}'_{/\otimes} = (\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{k}')^{\times}, \tag{34}$$

which shows that the current oriento-position gradient $\boldsymbol{A}'_{/\otimes} = \mathrm{d}\boldsymbol{A}'/\mathrm{d}\boldsymbol{x}$ is a third-order dual tensor characterized by the second-order dual tensor

$$\boldsymbol{k}' = \boldsymbol{A}'\mathrm{ax}(\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{A}'_{/\otimes}), \tag{35}$$

referred to as the *deformed curvature* $\boldsymbol{k}' = \boldsymbol{k}'_j \otimes \boldsymbol{g}^j$. This has a dual explicit form

$$\boldsymbol{k}' = \boldsymbol{k}'_{\mathrm{a}} + \varepsilon\boldsymbol{k}'_{\mathrm{l}} = \boldsymbol{\alpha}'\mathrm{ax}(\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{\alpha}'_{/\otimes}) + \varepsilon\boldsymbol{\alpha}'(\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{x}')_{/\otimes} \tag{36}$$

made of the deformed angular curvature and of the corotational gradient of the current position vector. Instead, the self-based version of the deformed curvature, i.e.

$$\boldsymbol{X}'^{\mathrm{T}}\boldsymbol{k}' = \boldsymbol{k}'_{\mathrm{a}} + \varepsilon(\boldsymbol{k}'_{\mathrm{l}} - \boldsymbol{x}' \times \boldsymbol{k}'_{\mathrm{a}}) = \boldsymbol{\alpha}'\mathrm{ax}(\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{\alpha}'_{/\otimes}) + \varepsilon\boldsymbol{x}'_{/\otimes}, \tag{37}$$

has the deformed position gradient $\boldsymbol{x}'_{/\otimes} = \mathrm{d}\boldsymbol{x}'/\mathrm{d}\boldsymbol{x}$ as dual part, namely the deformation gradient $\boldsymbol{F} = \boldsymbol{g}'_j \otimes \boldsymbol{g}^j$.

In a variational context, we also have to consider variations of the current configuration, in particular the virtual and incremental oriento-position variations $\delta\boldsymbol{A}'$ and $\partial\boldsymbol{A}'$. By observing that the gradient $\boldsymbol{A}'_{/\otimes}$ represents the spatial variation of the current oriento-position, we can see that we are actually dealing with three independent variations. Now, due to the inherent non-linearity of the representation of rototranslations, the virtual and incremental variations are unavoidably coupled, and this entails the presence of mixed virtual-incremental variations. To summarize, seven different variations of the configuration field must be considered independently, namely the virtual, incremental and mixed virtual-incremental variations $\delta\boldsymbol{A}'$, $\partial\boldsymbol{A}'$, $\partial\delta\boldsymbol{A}'$, the gradient $\boldsymbol{A}'_{/\otimes}$, and the virtual, incremental and mixed virtual-incremental variations of the gradient, $\delta\boldsymbol{A}'_{/\otimes}$, $\partial\boldsymbol{A}'_{/\otimes}$, $\partial\delta\boldsymbol{A}'_{/\otimes}$. According to the differentiation of the rototranslation discussed in Section 3, this leads to define, besides the finite curvature dual tensor $\boldsymbol{k}'$, a total of six virtual, incremental and mixed virtual-incremental characteristic differential dual vectors and tensors. In fact, by writing Eqs. (27) with any possible combination of variations, we obtain Eq. (34) and the following formulae,

$$\begin{aligned}
&\boldsymbol{A}'^{\mathrm{T}}\delta\boldsymbol{A}' = (\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{a}'_{\delta})^{\times},\\
&\boldsymbol{A}'^{\mathrm{T}}\partial\boldsymbol{A}' = (\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{a}'_{\partial})^{\times},\\
&\boldsymbol{A}'^{\mathrm{T}}\partial\delta\boldsymbol{A}' = (\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{a}'_{\partial\delta})^{\times} + \frac{1}{2}((\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{a}'_{\delta}) \times (\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{a}'_{\partial})^{\times} + (\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{a}'_{\partial}) \times (\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{a}'_{\delta})^{\times}),
\end{aligned} \tag{38}$$

$$\begin{aligned}
&\boldsymbol{A}'^{\mathrm{T}}\delta\boldsymbol{A}'_{/\otimes} = (\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{k}'_{\delta})^{\times} - \frac{1}{2}(\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{a}'_{\delta} \times \boldsymbol{k}')^{\times} + (\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{a}'_{\delta}) \times (\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{k}')^{\times},\\
&\boldsymbol{A}'^{\mathrm{T}}\partial\boldsymbol{A}'_{/\otimes} = (\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{k}'_{\partial})^{\times} - \frac{1}{2}(\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{a}'_{\partial} \times \boldsymbol{k}')^{\times} + (\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{a}'_{\partial}) \times (\boldsymbol{A}'^{T}\boldsymbol{k}')^{\times},\\
&\boldsymbol{A}'^{\mathrm{T}}\partial\delta\boldsymbol{A}'_{/\otimes} = (\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{k}'_{\partial\delta})^{\times} - \frac{1}{2}(\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{a}'_{\partial\delta} \times \boldsymbol{k}')^{\times} + (\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{a}'_{\partial\delta}) \times (\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{k}')^{\times}\\
&\qquad - \frac{1}{2}(\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{a}'_{\delta} \times \boldsymbol{k}'_{\partial} + \boldsymbol{A}'^{\mathrm{T}}\boldsymbol{a}'_{\partial} \times \boldsymbol{k}'_{\delta})^{\times} + (\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{a}'_{\delta}) \times (\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{k}'_{\partial})^{\times} + (\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{a}'_{\partial}) \times (\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{k}'_{\delta})^{\times},
\end{aligned} \tag{39}$$

which define the *virtual, incremental and mixed virtual-incremental oriento-position dual vectors* $\boldsymbol{a}'_\delta$, $\boldsymbol{a}'_\partial$, $\boldsymbol{a}'_{\partial\delta}$, *and curvature dual tensors* $\boldsymbol{k}'_\delta$, $\boldsymbol{k}'_\partial$, $\boldsymbol{k}'_{\partial\delta}$, as characteristic axial vectors and tensors.

Once again, it is worth studying the explicit form of the first differential oriento-position dual vectors $\boldsymbol{a}'_\delta$ and $\boldsymbol{a}'_\partial$. For the virtual differential vector, for instance, from Eqs. (38) and (28) it follows that

$$\boldsymbol{a}'_\delta = \boldsymbol{a}'_{a\delta} + \varepsilon\, \boldsymbol{a}'_{l\delta} = \boldsymbol{\alpha}' \mathrm{ax}(\boldsymbol{\alpha}'^{\mathrm{T}} \delta\boldsymbol{\alpha}') + \varepsilon\, \boldsymbol{\alpha}' \delta(\boldsymbol{\alpha}'^{\mathrm{T}} \boldsymbol{x}'). \tag{40}$$

The angular and linear parts are the virtual orientation vector and the corotational virtual variation of the current position vector, respectively. The relevant self-based version,

$$\boldsymbol{X}'^{\mathrm{T}} \boldsymbol{a}'_\delta = \boldsymbol{a}'_{a\delta} + \varepsilon\, (\boldsymbol{a}'_{l\delta} - \boldsymbol{x}' \times \boldsymbol{a}'_{a\delta}) = \boldsymbol{\alpha}' \mathrm{ax}(\boldsymbol{\alpha}'^{\mathrm{T}} \delta\boldsymbol{\alpha}') + \varepsilon\, \delta\boldsymbol{x}', \tag{41}$$

has the plain virtual variation of the current position vector as dual part. The explicit forms in Eqs. (40) and (41) hold also for the incremental differential vector $\boldsymbol{a}'_\partial$, and show that the pole-based linearization of the solution process will produce a 'corotational' tangent and give rise to a helicoidal modeling also in the solution space, with the particle orientations contributing to define the particle trajectories along the deformation history. Of course, the updating technique of the oriento-positions must be consistent with the helicoidal modeling along the solution process.

A number of 'corototranslational' variation formulae of the characteristic axial vectors and tensors can be obtained as for Eqs. (26) (Merlini and Morandini, 2002). The most useful ones are given here for future reference:

$$\begin{aligned} \boldsymbol{A}' \partial(\boldsymbol{A}'^{\mathrm{T}} \boldsymbol{a}'_\delta) &= \delta\boldsymbol{a}'_\partial = \boldsymbol{a}'_{\partial\delta} - \frac{1}{2} \boldsymbol{a}'_\partial \times \boldsymbol{a}'_\delta \\ \boldsymbol{A}' \partial(\boldsymbol{A}'^{\mathrm{T}} \boldsymbol{k}') &= \boldsymbol{a}'_{\partial/\otimes} = \boldsymbol{k}'_\partial - \frac{1}{2} \boldsymbol{a}'_\partial \times \boldsymbol{k}' \\ \boldsymbol{A}' \partial\delta(\boldsymbol{A}'^{\mathrm{T}} \boldsymbol{k}') &= \boldsymbol{k}'_{\partial\delta} - \frac{1}{2} (\boldsymbol{a}'_\partial \times \boldsymbol{k}'_\delta + \boldsymbol{a}'_\delta \times \boldsymbol{k}'_\partial + \boldsymbol{a}'_{\partial\delta} \times \boldsymbol{k}') + \frac{1}{2} (\boldsymbol{a}'_\partial \otimes \boldsymbol{a}'_\delta + \boldsymbol{a}'_\delta \otimes \boldsymbol{a}'_\partial) \cdot \boldsymbol{k}'. \end{aligned} \tag{42}$$

4.5. The rototranslation field

Instead of the current configuration, the transformation from the known reference configuration to the current one is often assumed as the problem unknown. In the case of helicoidal modeling, this can be done by putting

$$\boldsymbol{A}' = \boldsymbol{H}\boldsymbol{A}, \tag{43}$$

where $\boldsymbol{H}(\boldsymbol{x})$ is the origin-based *rototranslation field*. Eq. (43) is the biunique relation between the two kinds of unknowns (Fig. 2). Application of Eqs. (21) to the rototranslation $\boldsymbol{H} = \boldsymbol{A}'\boldsymbol{A}^{\mathrm{T}}$ between the oriento-positions defined by Eqs. (28) and (33), leads to the following expressions,

$$\begin{aligned} \boldsymbol{H} &= \boldsymbol{X}' \boldsymbol{\Phi} \boldsymbol{X}^{\mathrm{T}} \\ &= (\boldsymbol{I} + \varepsilon\, (\boldsymbol{x}' - \boldsymbol{\Phi}\boldsymbol{x}) \times) \boldsymbol{\Phi}, \end{aligned} \tag{44}$$

where $\boldsymbol{x}' - \boldsymbol{\Phi}\boldsymbol{x}$ is the *translation vector* (a vector far different from the displacement $\boldsymbol{u} = \boldsymbol{x}' - \boldsymbol{x}$, see Eq. $(22)_1$) and $\boldsymbol{\Phi} = \boldsymbol{\alpha}'\boldsymbol{\alpha}^{\mathrm{T}}$ the *rotation tensor*.

Spatial derivatives of the rototranslation tensor with respect to the initial coordinates yield equation

$$\boldsymbol{H}^{\mathrm{T}} \boldsymbol{H}_{/\otimes} = (\boldsymbol{H}^{\mathrm{T}} \boldsymbol{\omega})^{\times}, \tag{45}$$

showing that the rototranslation gradient $\boldsymbol{H}_{/\otimes} = \mathrm{d}\boldsymbol{H}/\mathrm{d}\boldsymbol{x}$ is a third-order dual tensor characterized by the second-order dual tensor

$$\boldsymbol{\omega} = \boldsymbol{H}\,\mathrm{ax}(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{H}_{/\otimes}), \tag{46}$$

which can be called the (*generalized*) *strain* tensor $\boldsymbol{\omega} = \boldsymbol{\omega}_j \otimes \boldsymbol{g}^j$ (cf. Eqs. (34) and (35)). The relevant explicit form, as obtained by deriving Eqs. (44),

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{\mathrm{a}} + \varepsilon\boldsymbol{\omega}_{\mathrm{l}} = \boldsymbol{\Phi}\,\mathrm{ax}(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi}_{/\otimes}) + \varepsilon\boldsymbol{\Phi}(\boldsymbol{\Phi}^{\mathrm{T}}(\boldsymbol{x}' - \boldsymbol{\Phi}\boldsymbol{x}))_{/\otimes}, \tag{47}$$

is made of the angular strain and the corotational gradient of the translation vector. The self-based version, instead,

$$\boldsymbol{X}'^{\mathrm{T}}\boldsymbol{\omega} = \boldsymbol{\omega}_{\mathrm{a}} + \varepsilon(\boldsymbol{\omega}_{\mathrm{l}} - \boldsymbol{x}' \times \boldsymbol{\omega}_{\mathrm{a}}) = \boldsymbol{\Phi}\,\mathrm{ax}(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi}_{/\otimes}) + \varepsilon(\boldsymbol{x}'_{/\otimes} - \boldsymbol{\Phi}\boldsymbol{x}_{/\otimes}), \tag{48}$$

has the 'linear' strain $\boldsymbol{\chi} = \boldsymbol{F} - \boldsymbol{\Phi}\boldsymbol{I}$ as dual part, i.e. the counterpart of the angular strain $\boldsymbol{\omega}_{\mathrm{a}} = \boldsymbol{k}'_{\mathrm{a}} - \boldsymbol{\Phi}\boldsymbol{k}_{\mathrm{a}}$ in the classical modeling (Merlini, 1997).

By taking the spatial derivatives of Eq. (43), the relation between the gradients of the current and reference oriento-positions is obtained as $\boldsymbol{A}'_{/\otimes} = \boldsymbol{A}'(\boldsymbol{A}'^{T}\boldsymbol{\omega})^{\times} + \boldsymbol{H}\boldsymbol{A}_{/\otimes}$. Then, the relation between curvatures and strain follows immediately from Eqs. (34) and (29),

$$\boldsymbol{k}' = \boldsymbol{\omega} + \boldsymbol{H}\boldsymbol{k}. \tag{49}$$

As far as the virtual and incremental variations are concerned, as for Eqs. (38) and (39), application of Eqs. (27) yields the formulae for the variations of the rototranslation and its gradient, with the definition of the relevant characteristic differential dual vectors and tensors. The differential helices $\boldsymbol{\eta}_{\delta}$, $\boldsymbol{\eta}_{\partial}$ and $\boldsymbol{\eta}_{\partial\delta}$ will obviously coincide with the differential oriento-position dual vectors $\boldsymbol{a}'_{\delta}$, $\boldsymbol{a}'_{\partial}$ and $\boldsymbol{a}'_{\partial\delta}$, respectively, while appropriate relations are found among the other characteristic differentials relevant to the rototranslation and the oriento-position fields. We do not give here such relations and the related formulae of corototranslational variation (see Merlini and Morandini, 2002). However, it is worth noting again that the linear parts of the first differential helices $\boldsymbol{\eta}_{\delta}$ and $\boldsymbol{\eta}_{\partial}$ are just the corotational variations $\boldsymbol{\rho}_{\delta} = \boldsymbol{\Phi}\delta(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{x}')$ and $\boldsymbol{\rho}_{\partial} = \boldsymbol{\Phi}\partial(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{x}')$ of the current position vector, as discussed above in Section 4.2.

4.6. Strain

A rigid displacement preserves the mutual distances and orientations of all the particles within a body. Let us consider the oriento-positions $\boldsymbol{A} = (\boldsymbol{I} + \varepsilon\boldsymbol{x}\times)\boldsymbol{\alpha}$ and $\boldsymbol{A}^* = (\boldsymbol{I} + \varepsilon\boldsymbol{x}^*\times)\boldsymbol{\alpha}^*$ of two particles of a body, and the relevant rototranslations to new oriento-positions $\boldsymbol{A}' = (\boldsymbol{I} + \varepsilon\boldsymbol{x}'\times)\boldsymbol{\alpha}'$ and $\boldsymbol{A}^{*'} = (\boldsymbol{I} + \varepsilon\boldsymbol{x}^{*'}\times)\boldsymbol{\alpha}^{*'}$, respectively. It is easily seen that if $\boldsymbol{\alpha}' = \boldsymbol{\Phi}\boldsymbol{\alpha}$, $\boldsymbol{\alpha}^{*'} = \boldsymbol{\Phi}\boldsymbol{\alpha}^*$, and $\boldsymbol{x}^{*'} - \boldsymbol{x}' = \boldsymbol{\Phi}(\boldsymbol{x}^* - \boldsymbol{x})$, i.e. if the displacement is rigid, the two rototranslations $\boldsymbol{A}'\boldsymbol{A}^{\mathrm{T}}$ and $\boldsymbol{A}^{*'}\boldsymbol{A}^{*T}$ are represented by the same dual tensor $\boldsymbol{H} = \boldsymbol{\Phi} + \varepsilon(\boldsymbol{x}' - \boldsymbol{\Phi}\boldsymbol{x}) \times \boldsymbol{\Phi} = \boldsymbol{\Phi} + \varepsilon(\boldsymbol{x}^{*'} - \boldsymbol{\Phi}\boldsymbol{x}^*) \times \boldsymbol{\Phi}$, and vice-versa. Therefore, a unique rototranslation of all the particles of a body characterizes a rigid displacement.

By taking the derivatives of Eq. (43) for a rigid displacement, it follows that $\boldsymbol{A}'_{/\otimes} = \boldsymbol{H}\boldsymbol{A}_{/\otimes}$. Then, by using Eqs. (29) and (34), it is seen that $\boldsymbol{k}' = \boldsymbol{H}\boldsymbol{k}$, namely also the curvatures undergo the same unique rototranslation if the displacement is rigid. As a consequence, the difference

$$\boldsymbol{\omega} = \boldsymbol{k}' - \boldsymbol{H}\boldsymbol{k}, \tag{50}$$

(cf. Eq. (49)) appears as the appropriate *kinematical measure of the strain* within a continuum. It is worth noting that an appropriate difference of curvatures defines the strain measure, and the latter coincides with tensor $\boldsymbol{\omega}$ defined in Eq. (45) as the second-order dual tensor characterizing the rototranslation gradient.

5. Mechanics of polar continuum

By using the foregoing kinematics, we draw in this section the variational mechanics for finite elasticity of the polar medium, i.e. the medium capable of withstanding angular strains by opposing couple-stresses. The fundamentals of polar elasticity date back to the early sixties, when a number of remarkable papers were delivered within the framework of the Cosserat mechanics (Ericksen and Truesdell, 1958; Toupin, 1962, 1964; Mindlin and Tierstein, 1962; Mindlin, 1964; Eringen, 1966). Thereafter, we quote the milestone work by Kafadar and Eringen (1971), some papers dealing with variational formulations (Germain, 1973; Zubov, 1990; Felippa, 1992) and an application of micropolar elasticity to shell modeling (Yeh and Chen, 1993). Recent contributions to the mechanics of polar media are found in the papers by Le and Stumpf (1998), Nikitin and Zubov (1998), Pettinger and Abeyaratne (2000), Grekova and Zhilin (2001) and Yang et al. (2002).

In this work, we keep to a variational scheme ready for classical modeling (Merlini, 1997), and we arrange it for the helicoidal modeling. The variational formulation yields the main principles for elastostatics in a form retaining the boundary compatibility condition for a weak statement of the external constraints. After consistent linearization, the dependency of the linearized virtual functionals on the virtual, incremental and mixed virtual-incremental fields is stressed as a peculiar feature to the helicoidal modeling. In this general formulation for a polar continuum we can accommodate the non-polar medium as a particular case; in such case, the role of the orientation of the particles becomes related to the polar decomposition of the deformation gradient (see Part III).

5.1. Stress and equilibrium

We denote with $\boldsymbol{f}$ and $\boldsymbol{c}$ the densities of the external forces and couples acting in the current configuration within the body (per unit initial volume), and with $\boldsymbol{t}$ and $\boldsymbol{m}$ those acting on the boundary (per unit initial surface). (The surface traction $\boldsymbol{t}$ should be not confused with the translation vector in Sections 2.5 and 2.6) Forces and couples are also the primal and dual parts of self-based dual vectors, respectively, which can be conveniently transformed into consistent pole-based dual vectors by means of the arm $\boldsymbol{X}'$. This means taking the moments with respect to the pole (the origin in our case) and yields the *pole-based body* and *surface loads*

$$\boldsymbol{b} = \boldsymbol{X}'(\boldsymbol{f} + \varepsilon\boldsymbol{c}) = \boldsymbol{f} + \varepsilon(\boldsymbol{c} + \boldsymbol{x}' \times \boldsymbol{f}), \tag{51}$$

$$\boldsymbol{s} = \boldsymbol{X}'(\boldsymbol{t} + \varepsilon\boldsymbol{m}) = \boldsymbol{t} + \varepsilon(\boldsymbol{m} + \boldsymbol{x}' \times \boldsymbol{t}). \tag{52}$$

The stress and couple-stress tensors, analogs of the first Piola–Kirchhoff stress (Zubov, 1990), are denoted as $\tilde{\boldsymbol{T}}$ and $\tilde{\boldsymbol{M}}$, and are respectively the primal and dual parts of a self-based dual tensor. Transforming the latter into an origin-based dual tensor yields

$$\tilde{\boldsymbol{S}} = \boldsymbol{X}'(\tilde{\boldsymbol{T}} + \varepsilon\tilde{\boldsymbol{M}}) = \tilde{\boldsymbol{T}} + \varepsilon(\tilde{\boldsymbol{M}} + \boldsymbol{x}' \times \tilde{\boldsymbol{T}}). \tag{53}$$

$\tilde{\boldsymbol{S}}$ can be referred to as the *pole-based* first Piola–Kirchhoff dual stress tensor. The primal and dual parts of the load vectors and stress tensor can also be properly referred to as the *linear* and *angular parts*, respectively.

In terms of pole-based load densities and stress, the static balance equations within the volume V and at the body surface S are cast in an essential and very meaningful form,

$$\begin{aligned} &(\text{in } V) \quad \tilde{\boldsymbol{S}}_{/\bullet} + \boldsymbol{b} = \boldsymbol{0}, \\ &(\text{at } S) \quad \tilde{\boldsymbol{S}}\boldsymbol{\nu} = \boldsymbol{s}, \end{aligned} \tag{54}$$

where the notation of inner derivative $(\ )_{/\bullet}$ stands for div( ), and $\boldsymbol{\nu}$ is the outward unit normal. Eqs. (54) are the *pole-based dual equilibrium equations*; the relevant self-based equations are easily retrieved in the usual linear and angular forms, $\tilde{\boldsymbol{T}}_{/\bullet} + \boldsymbol{f} = \boldsymbol{0}$ and $\tilde{\boldsymbol{M}}_{/\bullet} + 2\mathrm{ax}(\tilde{\boldsymbol{T}}\boldsymbol{F}^{\mathrm{T}}) + \boldsymbol{c} = \boldsymbol{0}$ in V, and $\tilde{\boldsymbol{T}}\boldsymbol{\nu} = \boldsymbol{t}$ and $\tilde{\boldsymbol{M}}\boldsymbol{\nu} = \boldsymbol{m}$ at S (cf. Malvern, 1969).

5.2. Dual spaces and scalar product

Dual vectors and tensors discussed so far, clearly belong to either the kinematic space or its dual space of co-vectors (loads and stresses). Such well-distinct vector spaces in mechanics correspond to the spaces of covariant vectors and contravariant vectors in tensor algebra (Bowen and Wang, 1976; Ogden, 1984). The *scalar product* between tensors of the two spaces is of primary importance for a variational formulation. Let us denote with $\boldsymbol{A} = \boldsymbol{A}_{\mathrm{a}} + \varepsilon \boldsymbol{A}_{\mathrm{l}}$ and $\boldsymbol{B} = \boldsymbol{B}_{\mathrm{l}} + \varepsilon \boldsymbol{B}_{\mathrm{a}}$ two dual tensors of the same order (whatever it is, i.e. vectors, second-order tensors, or higher-order tensors), belonging respectively to the kinematic and co-kinematic spaces. Notice that the subscripts denoting the 'angular' and 'linear' parts are consistent with the physical meaning of each single part of both kinds of tensors, but in an opposite sense: in fact, the dual part, which is dimensionally greater by a length than the primal part, is the linear part for kinematic tensors and the angular part (the moment) for co-kinematic tensors. With reference to the usual angle bracket $\langle , \rangle$ notation in tensor algebra, their scalar product ought to be computed in such a way that

$$\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \langle \boldsymbol{A}_{\mathrm{a}}, \boldsymbol{B}_{\mathrm{a}} \rangle + \langle \boldsymbol{A}_{\mathrm{l}}, \boldsymbol{B}_{\mathrm{l}} \rangle, \tag{55}$$

yielding a scalar-valued real quantity, dimensionally consistent with a mechanical work-related quantity. This must hold true independently of the order of tensors, so the scalar products on the right-hand-side of Eq. (55) must correspond to full contractions between real tensors.

It should be noted that an inner product, as for instance $\boldsymbol{A} : \boldsymbol{B}$ in the case of second-order tensors, cannot be directly substituted for $\langle \boldsymbol{A}, \boldsymbol{B} \rangle$ in Eq. (55). In fact, if one computes $\boldsymbol{A} : \boldsymbol{B}$ by the algebraic rules of dual numbers, he obtains a dual scalar far different from $\langle \boldsymbol{A}, \boldsymbol{B} \rangle$. In order to convert the scalar product defined by Eq. (55) into appropriate 'dot' products inside one vector space, it is expedient to arrange in columns the angular and linear parts of the dual tensors, taking care of keeping the same order for both dual spaces, e.g. $\{\boldsymbol{A}\} = [\boldsymbol{A}_{\mathrm{a}} \quad \boldsymbol{A}_{\mathrm{l}}]^{\mathrm{T}}$ and $\{\boldsymbol{B}\} = [\boldsymbol{B}_{\mathrm{a}} \quad \boldsymbol{B}_{\mathrm{l}}]^{\mathrm{T}}$. Then, on use of the rules of matrix algebra, the scalar product is straightforwardly resolved as $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \{\boldsymbol{a}\}^{\mathrm{T}} \cdot \{\boldsymbol{b}\}$, $\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \{\boldsymbol{A}\}^{\mathrm{T}} : \{\boldsymbol{B}\}$, and so on, yielding single-dot products between vectors, double-dot products between second-order tensors, ... n-dot products between nth-order tensors. The brackets are of course necessary and allow to distinguish between products like $\{\boldsymbol{A}\}^{\mathrm{T}} : \{\boldsymbol{B}\}$ and $\boldsymbol{A} : \boldsymbol{B}$.

5.3. Elasticity and constitutive equation

The constitutive equations for polar elasticity were obtained by Kafadar and Eringen (1971) from the conservation of energy. In this work we consider only hyperelastic materials, so we ignore any thermal effects and assume that the elastic potential is a function of the strain alone (Malvern, 1969). The constitutive equations can then be derived in a purely mechanical manner, e.g. from a balance of powers as in Grekova and Zhilin (2001). We keep to the static case and use a balance of virtual works, as in Merlini (1997). In terms of dual tensors, the formulation is as follows.

Enforcing the scalar product of the balance Eqs. (54) by a virtual oriento-position vector $\boldsymbol{a}'_{\delta}$ yields two scalar statements of the equilibrium in terms of virtual works: $\langle \boldsymbol{a}'_{\delta}, \tilde{\boldsymbol{S}}_{/\bullet} \rangle + \langle \boldsymbol{a}'_{\delta}, \boldsymbol{b} \rangle = 0$ in V and $-\langle \boldsymbol{a}'_{\delta}, \tilde{\boldsymbol{S}}\boldsymbol{\nu} \rangle + \langle \boldsymbol{a}'_{\delta}, \boldsymbol{s} \rangle = 0$ at S. By using the identity $\{\boldsymbol{a}'_{\delta}\}^{\mathrm{T}} \cdot \{\tilde{\boldsymbol{S}}_{/\bullet}\} = (\{\boldsymbol{a}'_{\delta}\}^{\mathrm{T}} \cdot \{\tilde{\boldsymbol{S}}\})_{/\bullet} - \{\boldsymbol{a}'_{\delta/\otimes}\}^{\mathrm{T}} : \{\tilde{\boldsymbol{S}}\}$, recalling Eq. $(42)_2$ and observing that $\boldsymbol{A}'\delta(\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{k}') = \boldsymbol{H}\delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}) = \boldsymbol{H}\boldsymbol{X}\delta(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega})$, such statements take the form

$$\begin{aligned}&(\text{in } V) \quad (\{\boldsymbol{a}'_\delta\}^{\mathrm{T}} \cdot \{\tilde{\boldsymbol{S}}\})_{/\bullet} - \langle \delta(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}), \boldsymbol{X}\boldsymbol{H}^{\mathrm{T}}\tilde{\boldsymbol{S}}\rangle + \langle \boldsymbol{a}'_\delta, \boldsymbol{b}\rangle = 0,\\ &(\text{at } S) \quad -\langle \boldsymbol{a}'_\delta, \tilde{\boldsymbol{S}}\boldsymbol{v}\rangle + \langle \boldsymbol{a}'_\delta, \boldsymbol{s}\rangle = 0.\end{aligned} \tag{56}$$

The former equation is a statement of the principle of virtual work written for an elementary body volume. The first term reflects the virtual work of the surface loads, as evidenced by the second equation. Therefore, the scalar product $\langle \delta(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}), \boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}\tilde{\boldsymbol{S}}\rangle$ represents the virtual work of the internal actions, per unit reference volume. Under the assumption of *hyperelasticity*, a *strain-energy* density function $w(\boldsymbol{\xi})$ of some *strain parameter* $\boldsymbol{\xi}$ exists, and its derivative with respect to the strain parameter itself determines (according to the differentiation rule $\mathrm{d}w = \langle \mathrm{d}\boldsymbol{\xi}, w_{/\boldsymbol{\xi}}\rangle$) the work-conjugate stress parameter $\hat{\boldsymbol{S}}(\boldsymbol{\xi})$ as $w_{/\boldsymbol{\xi}}$. The virtual variation δw of the strain energy coincides with the work of the stresses by the virtual variation of the corresponding strains, namely with the second term in Eq. $(56)_1$. This entails that actually w is *not* a direct function of the kinematical strain measure $\boldsymbol{\omega}$ from Eq. (50); instead, it is a function of tensor $\boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}$ as *strain parameter*, which represents the self-based back-rototranslated version of tensor $\boldsymbol{\omega}$. Conversely, the *stress parameter* $w_{/\boldsymbol{\xi}}$ is *not* the first Piola–Kirchhoff dual stress tensor $\tilde{\boldsymbol{S}}$, but its self-based back-rototranslated version, tensor $\boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}\tilde{\boldsymbol{S}}$.

By recalling Eq. $(44)_1$, such strain and stress parameters can also be written $\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{X}'^{\mathrm{T}}\boldsymbol{\omega}$ and $\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{X}'^{\mathrm{T}}\tilde{\boldsymbol{S}}$, respectively. This changes their meaning from self-basing a back-rototranslated, pole-based strain or stress to back-rotating a self-based strain or stress, cf. Eqs. (48) and (53). Anyway, the argument of which is function the (pole-independent) strain energy, and the first derivative itself of the strain energy, become identified in a natural way by the self-based, pole-independent second-order dual tensors

$$\begin{aligned}\boldsymbol{\xi} &= \boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega} = \boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{X}'^{\mathrm{T}}\boldsymbol{\omega}\\ \hat{\boldsymbol{S}} &= \boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}\tilde{\boldsymbol{S}} = \boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{X}'^{\mathrm{T}}\tilde{\boldsymbol{S}}.\end{aligned} \tag{57}$$

The relevant explicit forms, for Eqs. (48) and (53), read

$$\begin{aligned}\boldsymbol{\xi} &= \boldsymbol{\beta} + \varepsilon\boldsymbol{\varepsilon} = \boldsymbol{\Phi}^{\mathrm{T}}(\boldsymbol{\omega}_{\mathrm{a}} + \varepsilon\boldsymbol{\chi}) = \mathrm{ax}(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi}_{/\otimes}) + \varepsilon\boldsymbol{\Phi}^{\mathrm{T}}(\boldsymbol{x}'_{/\otimes} - \boldsymbol{\Phi}\boldsymbol{x}_{/\otimes}),\\ \hat{\boldsymbol{S}} &= \hat{\boldsymbol{T}} + \varepsilon\hat{\boldsymbol{M}} = \boldsymbol{\Phi}^{\mathrm{T}}(\tilde{\boldsymbol{T}} + \varepsilon\tilde{\boldsymbol{M}}),\end{aligned} \tag{58}$$

and consist of the strain and stress parameters of the Biot kind, as formerly used with classical modeling (Merlini, 1997). In particular, the real tensors $\boldsymbol{I} + \boldsymbol{\varepsilon}$ and $\boldsymbol{\beta}$ are just the Cosserat deformation tensor and the wryness tensor introduced by Kafadar and Eringen (1971) as the appropriate strain measures yielding constitutive equations that intrinsically satisfy the principle of objectivity for polar media. So, the proposed functional relationship $w(\boldsymbol{\xi})$ is frame indifferent (see also Nikitin and Zubov, 1998; Le and Stumpf, 1998).

The relation

$$\hat{\boldsymbol{S}} = w_{/\boldsymbol{\xi}}, \tag{59}$$

giving the stress-parameter function $\hat{\boldsymbol{S}}(\boldsymbol{\xi})$, represents the (direct) *constitutive equation* of the polar continuum. It corresponds to the constitutive equations obtained by Kafadar and Eringen (1971), which would read $\tilde{\boldsymbol{T}} = \boldsymbol{\Phi}w_{/\boldsymbol{\varepsilon}}$ and $\tilde{\boldsymbol{M}} = \boldsymbol{\Phi}w_{/\boldsymbol{\beta}}$ in terms of the analogs of the first Piola–Kirchhoff stress tensor, cf. Eqs. (53) and $(57)_2$. The tangent map ensuing from the linearization of the stress-parameter function is a fourth-order tensor able to give a contravariant dual tensor when operating on a covariant one. In matrix notation, the linearized stress-strain elastic law takes the form

$$\{\partial\hat{\boldsymbol{S}}\} = [\mathbb{E}] : \{\partial\boldsymbol{\xi}\}, \tag{60}$$

where $[\mathbb{E}(\boldsymbol{\xi})] = [\hat{\boldsymbol{S}}_{/\boldsymbol{\xi}}] = [w_{/\boldsymbol{\xi}\boldsymbol{\xi}}]$ is a 2×2 symmetric and invertible matrix of fourth-order real tensors mapping $\{\partial\boldsymbol{\xi}\}$ on $\{\partial\hat{\boldsymbol{S}}\}$, for instance

$$[\mathbb{E}] = \begin{bmatrix} \mathbb{E}_{\mathrm{aa}} & \mathbb{E}_{\mathrm{al}} \\ \mathbb{E}_{\mathrm{la}} & \mathbb{E}_{\mathrm{ll}} \end{bmatrix} \tag{61}$$

with the arrangement sequence 'angular-linear'. The operator $[\mathbb{E}]$ is shortly referred to as the *elastic tensor* of the polar continuum.

Solving the constitutive equation $\hat{\boldsymbol{S}}(\boldsymbol{\xi})$ for the strain parameter gives the inverse function $\boldsymbol{\xi}(\hat{\boldsymbol{S}})$, which allows introducing the *complementary-energy* density function $v(\hat{\boldsymbol{S}})$ by means of the Legendre transform $v = \langle \hat{\boldsymbol{S}}, \boldsymbol{\xi} \rangle - w$. The strain-parameter function $\boldsymbol{\xi}(\hat{\boldsymbol{S}})$ is recognized to be the derivative of the complementary energy with respect to the work-conjugate stress parameter, $\boldsymbol{\xi} = v_{/\hat{\boldsymbol{S}}}$. This equation represents the *inverse constitutive equation* of the polar continuum, to be used in mixed formulations.

5.4. Compatibility conditions

Once the current oriento-position is assumed to be a differentiable function of the material coordinates, because of Eqs. $(57)_1$ and (50), a compatible strain parameter is guaranteed by the strain-displacement relation

$$\boldsymbol{\xi} = \boldsymbol{X}^{\mathrm{T}} \boldsymbol{H}^{\mathrm{T}} \boldsymbol{\omega}. \tag{62}$$

Eq. (62) constitutes the *internal compatibility condition*.

On the body surface, where constraints are expected, the compatibility of the current unknown oriento-position $\boldsymbol{A}'$ with a boundary value $\boldsymbol{A}'_{\mathrm{b}}$ can be evaluated by comparing the rototranslations $\boldsymbol{H} = \boldsymbol{A}'\boldsymbol{A}^{\mathrm{T}}$ and $\boldsymbol{H}_{\mathrm{b}} = \boldsymbol{A}'_{\mathrm{b}}\boldsymbol{A}^{\mathrm{T}}$ from the reference oriento-position $\boldsymbol{A} \equiv \boldsymbol{A}_{\mathrm{b}}$. The equation must concern truly independent parameters, and we choose to equate the relevant logarithmic maps,

$$\log \boldsymbol{H} = \log \boldsymbol{H}_{\mathrm{b}}. \tag{63}$$

Eq. (63) constitutes the *boundary compatibility condition*.

5.5. Variational framework

A variational framework leading to the principles governing the polar continuum mechanics is established on the basis of the equilibrium Eqs. (54), the constitutive Eq. (59) and the compatibility Eqs. (62) and (63), by assuming as unknowns the parameters describing the current oriento-position $\boldsymbol{A}'$ (yet to be chosen), the strain parameter $\boldsymbol{\xi}$ and the stress parameter $\hat{\boldsymbol{S}}$. First, an equivalent scalar form of the equation set is written by means of scalar products with arbitrary dual tensor multipliers. By choosing as multipliers appropriate virtual quantities related to the unknowns themselves, all the scalar terms have the physical meaning of virtual work:

$$\begin{aligned} &(\text{in } V)\begin{cases} \langle \boldsymbol{a}'_{\delta}, (\boldsymbol{H}\boldsymbol{X}\hat{S})_{/\bullet} + \boldsymbol{b} \rangle = 0, \\ \langle \delta\boldsymbol{\xi}, \hat{\boldsymbol{S}} - w_{/\boldsymbol{\xi}} \rangle = 0, \\ \langle \delta\hat{\boldsymbol{S}}, \boldsymbol{\xi} - \boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega} \rangle = 0, \end{cases} \\ &(\text{at } S)\begin{cases} \langle \boldsymbol{a}'_{\delta,} - \boldsymbol{H}\boldsymbol{X}\hat{\boldsymbol{S}}\boldsymbol{v} + \boldsymbol{s} \rangle = 0, \\ \langle \delta(\boldsymbol{A}\boldsymbol{X}\hat{\boldsymbol{S}}\boldsymbol{v}), \mathrm{ax}(\log \boldsymbol{H} - \log \boldsymbol{H}_{\mathrm{b}}) \rangle = 0. \end{cases} \end{aligned} \tag{64}$$

Notice that, for the time being, we formally write the scalar boundary conditions for the whole surface of the body, regardless of possible differences between free/loaded portions and constrained portions. Of course, their meanings are quite different on the two kinds of boundary: on the free/loaded portions, the equilibrium is an equation to fulfill while the compatibility just provides the surface current configuration;

instead, on the constrained portions the compatibility is an equation to fulfill and the equilibrium provides the constraint reactions.

Following the standard schemes of the calculus of variations, Eqs. (64) are integrated over the relevant domains, then appropriate terms are transferred from volume to surface integrals by means of the theorem of the divergence. In this process, it is worth analyzing the development of terms $\langle \boldsymbol{a}'_{\delta}, (\boldsymbol{H}\boldsymbol{X}\hat{\boldsymbol{S}})_{/\bullet} \rangle$ and $-\langle \delta\hat{\boldsymbol{S}}, \boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega} \rangle$. Their sum can be written in one of the following equivalent forms:

$$
\begin{aligned}
&\langle \boldsymbol{a}'_{\delta}, (\boldsymbol{H}\boldsymbol{X}\hat{\boldsymbol{S}})_{/\bullet} \rangle - \langle \delta\hat{\boldsymbol{S}}, \boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega} \rangle \\
&\quad = \frac{1}{2}\langle \delta\boldsymbol{H}, (\boldsymbol{H}(\boldsymbol{X}\hat{\boldsymbol{S}})^{\times})_{/\bullet} \rangle - \frac{1}{2}\langle \delta(\boldsymbol{H}(\boldsymbol{X}\hat{\boldsymbol{S}})^{\times}), \boldsymbol{H}_{/\otimes} \rangle && \text{(65a)} \\
&\quad = (\{\boldsymbol{a}'_{\delta}\}^{\mathrm{T}} \cdot \{\boldsymbol{H}\boldsymbol{X}\hat{\boldsymbol{S}}\})_{/\bullet} - \delta\langle \hat{\boldsymbol{S}}, \boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega} \rangle && \text{(65b)} \\
&\quad = (\{\boldsymbol{a}'_{\delta}\}^{\mathrm{T}} \cdot \{\boldsymbol{H}\boldsymbol{X}\hat{\boldsymbol{S}}\} - \{\operatorname{ax}\log\boldsymbol{H}\}^{\mathrm{T}} \cdot \{\delta(\boldsymbol{\Lambda}\boldsymbol{X}\hat{\boldsymbol{S}})\})_{/\bullet} \\
&\qquad + \frac{1}{2}\langle \delta(\boldsymbol{\Lambda}\boldsymbol{X}\hat{\boldsymbol{S}})^{\times}_{/\bullet}, \log\boldsymbol{H} \rangle - \frac{1}{2}\langle \delta(\log\boldsymbol{H})_{/\otimes}, (\boldsymbol{\Lambda}\boldsymbol{X}\hat{\boldsymbol{S}})^{\times} \rangle. && \text{(65c)}
\end{aligned}
$$

The manipulations involved in Eqs. (65) are detailed in Merlini and Morandini (2002). We exploited: Eqs. (38), (42), (43), (45) and (50); the identities $\boldsymbol{a}\cdot\boldsymbol{b} = \frac{1}{2}\boldsymbol{a}\times : \boldsymbol{b}\times$, $\boldsymbol{a}\cdot\boldsymbol{B} = \frac{1}{2}\boldsymbol{a}\times : \boldsymbol{B}^{\times}$, $\boldsymbol{A} : \boldsymbol{B} = \frac{1}{2}\boldsymbol{A}^{\times} \vdots \boldsymbol{B}^{\times}$ about the inner products; the identities $(\boldsymbol{a}\times)_{/\otimes} = \boldsymbol{a}^{\times}_{/\otimes}$ and $(\boldsymbol{A}^{\times})_{/\bullet} = \boldsymbol{A}_{/\bullet}\times$ about gradient and divergence of skew-symmetric tensors; the identities $(\boldsymbol{a}\cdot\boldsymbol{B})_{/\bullet} = \boldsymbol{a}_{/\otimes} : \boldsymbol{B} + \boldsymbol{a}\cdot\boldsymbol{B}_{/\bullet}$ and $(\boldsymbol{A} : \mathscr{B})_{/\bullet} = \boldsymbol{A}_{/\otimes} \vdots \mathscr{B} + \boldsymbol{A} : \mathscr{B}_{/\bullet}$ about spatial derivatives; the definition $\boldsymbol{\Lambda} = \operatorname{dexp}\log\boldsymbol{H}$ of the differential map associated with the rototranslation $\boldsymbol{H}$; the property $\boldsymbol{H} = \boldsymbol{\Lambda}\boldsymbol{\Lambda}^{-\mathrm{T}} = \boldsymbol{\Lambda}^{-\mathrm{T}}\boldsymbol{\Lambda}$ relating the exponential and differential maps of the rototranslation; and the formula $\boldsymbol{\omega} = \boldsymbol{\Lambda}\operatorname{ax}(\log\boldsymbol{H})_{/\otimes}$ relating the strain with the spatial derivatives of the rototranslation logarithmic map.

In Eqs. (65), the original sum is rewritten as Eq. (65a) as the difference of two scalar products of virtual variations by spatial derivatives. The first of such scalar products is being transferred to the boundary surface in Eq. (65b), and both are being transferred to the boundary surface in Eq. (65c). The terms being transferred to the boundary surface are the scalar divergence of dual vectors in Eqs. (65b) and (65c). The remaining terms in all the three forms are volume terms and deserve further discussion. In Eq. (65b), such terms reduce to the virtual variation of the scalar product of stress $\tilde{\boldsymbol{S}} = \boldsymbol{H}\boldsymbol{X}\hat{\boldsymbol{S}}$ by strain $\boldsymbol{\omega}$, while in Eqs. (65a) and (65c) they have a symplectic structure. In Eq. (65a), the kinematic and stress variables are respectively the second-order dual tensor $\boldsymbol{H}$ and the third-order dual tensor $\boldsymbol{H}(\boldsymbol{X}\hat{\boldsymbol{S}})^{\times}$, and the scalar products are between the virtual variation of one variable and the spatial derivative of the other variable. In Eq. (65c), instead, the variables are the second-order dual tensor $\log\boldsymbol{H}$ and the third-order dual tensor $(\boldsymbol{\Lambda}\boldsymbol{X}\hat{\boldsymbol{S}})^{\times}$, and the scalar products are between the virtual variation of the spatial derivative of one variable and the other variable as is. Note that it has been possible to complete this picture, that proves the consistency of the choice of the conjugate virtual fields, thanks to the particular structure of the virtual multiplier of the boundary compatibility condition in Eqs. (64).

By using Eq. (65a), the *Weak Variational Form* is obtained,

$$
\begin{aligned}
&\int_V \left\{ \frac{1}{2}\langle \delta\boldsymbol{H}, (\boldsymbol{H}(\boldsymbol{X}\hat{\boldsymbol{S}})^{\times})_{/\bullet} \rangle - \frac{1}{2}\langle \delta(\boldsymbol{H}(\boldsymbol{X}\hat{\boldsymbol{S}})^{\times}), \boldsymbol{H}_{/\otimes} \rangle - \delta[w - \langle \boldsymbol{\xi}, \hat{\boldsymbol{S}} \rangle] + \langle \boldsymbol{a}'_{\delta}, \boldsymbol{b} \rangle \right\} \mathrm{d}V \\
&\quad + \int_S \{ \langle \boldsymbol{a}'_{\delta}, -\boldsymbol{H}\boldsymbol{X}\hat{\boldsymbol{S}}\boldsymbol{v} + \boldsymbol{s} \rangle + \langle \delta(\boldsymbol{\Lambda}\boldsymbol{X}\hat{\boldsymbol{S}}\boldsymbol{v}), \operatorname{ax}\log\boldsymbol{H} - \operatorname{ax}\log\boldsymbol{H}_{\mathrm{b}} \rangle \} \mathrm{d}S = 0, && \text{(66)}
\end{aligned}
$$

which is characterized by the absence of spatial derivatives in the virtual fields. By using Eq. (65b), the *Weaker Variational Form* is obtained,

$$\int_V \{ - \delta[w - \langle \hat{\boldsymbol{S}}, \boldsymbol{\xi} - \boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}\rangle] + \langle \boldsymbol{a}'_\delta, \boldsymbol{b}\rangle\}\mathrm{d}V$$
$$+ \int_S \{\langle \boldsymbol{a}'_\delta, \boldsymbol{s}\rangle + \langle \delta(\boldsymbol{\Lambda}\boldsymbol{X}\hat{\boldsymbol{S}}\boldsymbol{v}), \mathrm{ax}\log\boldsymbol{H} - \mathrm{ax}\log\boldsymbol{H}_{\mathrm{b}}\rangle\}\mathrm{d}S = 0, \tag{67}$$

which contains the virtual variation of the kinematical strain $\boldsymbol{\omega}$. Finally, using Eq. (65c), the *Weakest Variational Form* is obtained,

$$\int_V \left\{ \frac{1}{2}\langle \delta(\boldsymbol{\Lambda}\boldsymbol{X}\hat{\boldsymbol{S}})^{\times}_{/\bullet}, \log\boldsymbol{H}\rangle - \frac{1}{2}\langle \delta(\log\boldsymbol{H})_{/\otimes}, (\boldsymbol{\Lambda}\boldsymbol{X}\hat{\boldsymbol{S}})^{\times}\rangle - \delta\Big[w - \langle \boldsymbol{\xi}, \hat{\boldsymbol{S}}\rangle\Big] + \langle \boldsymbol{a}'_\delta, \boldsymbol{b}\rangle \right\}\mathrm{d}V$$
$$+ \int_S \left\{ \langle \boldsymbol{a}'_\delta, \boldsymbol{s}\rangle - \langle \delta(\boldsymbol{\Lambda}\boldsymbol{X}\hat{\boldsymbol{S}}\boldsymbol{v}), \mathrm{ax}\log\boldsymbol{H}_{\mathrm{b}}\rangle \right\}\mathrm{d}S = 0, \tag{68}$$

with the presence of spatial derivatives only within the virtual fields. As pointed out by Borri (1994) in analytical dynamics, the weakest form would entail, in a finite-element context, a continuity requirement for the trial functions one order less than for the test functions. Also the structure of the surface integrals in such variational forms is worth noting, going from a weak statement of the boundary conditions to the work of just the boundary loads $\boldsymbol{s}$ and rototranslations $\boldsymbol{H}_{\mathrm{b}}$.

The foregoing variational forms are based on the strain-energy density $w(\boldsymbol{\xi})$ and involve as unknowns *three* dual fields, $\boldsymbol{A}'$, $\boldsymbol{\xi}$ and $\hat{\boldsymbol{S}}$. They can be brought to forms based on the complementary-energy density $v(\hat{\boldsymbol{S}})$ by means of the Legendre transform $v = \langle \hat{\boldsymbol{S}}, \boldsymbol{\xi}\rangle - w$. In particular, the *Weaker Variational Form* from Eq. (67) is presented,

$$\int_V \left\{ \delta\Big[v - \langle \hat{\boldsymbol{S}}, \boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}\rangle\Big] + \langle \boldsymbol{a}'_\delta, \boldsymbol{b}\rangle \right\}\mathrm{d}V + \int_S \left\{ \langle \boldsymbol{a}'_\delta, \boldsymbol{s}\rangle + \langle \delta(\boldsymbol{\Lambda}\boldsymbol{X}\hat{\boldsymbol{S}}\boldsymbol{v}), \mathrm{ax}\log\boldsymbol{H} - \mathrm{ax}\log\boldsymbol{H}_{\mathrm{b}}\rangle \right\}\mathrm{d}S = 0. \tag{69}$$

The strain $\boldsymbol{\xi}$ disappears and the unknowns reduce to *two* dual fields, $\boldsymbol{A}'$ and $\hat{\boldsymbol{S}}$. The corresponding scalar equations are

$$(\text{in } V)\begin{cases} \langle \boldsymbol{a}'_\delta, (\boldsymbol{H}\boldsymbol{X}\hat{\boldsymbol{S}})_{/\bullet} + \boldsymbol{b}\rangle = 0, \\ \langle \delta\hat{\boldsymbol{S}}, v_{/\hat{\boldsymbol{S}}} - \boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}\rangle = 0, \end{cases} \tag{70}$$

within the volume, and again the scalar Eqs. $(64)_{4,5}$ at the boundary.

The formal variational framework discussed above yields the most important principles in elastostatics. Before stating them, however, it is worth discussing briefly the boundary terms. In fact, such terms allow taking into account the surface constraints in a weak manner. Since we will consider only the intermediate forms Eqs. (67) and (69), just the relevant boundary terms are discussed. Let us split the body surface into a free/loaded portion S_{f} and a constrained portion S_{c}. At S_{f} the compatibility condition can be discarded, for it is nothing but a trivial equation for retrieving the boundary rototranslation as $\boldsymbol{H}_{\mathrm{b}} = \boldsymbol{H}$, and just the term $\langle \boldsymbol{a}'_\delta, \boldsymbol{s}\rangle$ is integrated. At S_{c}, instead, the current oriento-position $\boldsymbol{A}'$ must comply with a known value $\boldsymbol{A}'_{\mathrm{c}} = \boldsymbol{H}_{\mathrm{c}}\boldsymbol{A}$, and the compatibility condition $\log\boldsymbol{H} = \log\boldsymbol{H}_{\mathrm{c}}$ is kept as a constraint equation to fulfill. The balance between stress and constraint reaction is used as a definition of the unknown reaction density as $\boldsymbol{s}_{\mathrm{c}} = \boldsymbol{H}\boldsymbol{X}\hat{\boldsymbol{S}}\boldsymbol{v}$, a pole-based dual vector with explicit expression $\boldsymbol{s}_{\mathrm{c}} = \boldsymbol{X}'(\boldsymbol{t}_{\mathrm{c}} + \varepsilon\boldsymbol{m}_{\mathrm{c}})$. So, $\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{s}_{\mathrm{c}}$ can be substituted for the multiplier $\boldsymbol{\Lambda}\boldsymbol{X}\hat{\boldsymbol{S}}\boldsymbol{v}$ and the relevant integrand term written as the scalar product $\langle \delta(\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{s}_{\mathrm{c}}), \mathrm{ax}\log\boldsymbol{H} - \mathrm{ax}\log\boldsymbol{H}_{\mathrm{c}}\rangle$. This term and the virtual work $\langle \boldsymbol{a}'_\delta, \boldsymbol{s}_{\mathrm{c}}\rangle$ of the reaction are integrated over $\boldsymbol{S}_{\mathrm{c}}$.

Therefore, the weaker variational forms, Eqs. (67) and (69), can be now written as

$$\int_V \delta\Big[w - \langle \hat{\boldsymbol{S}}, \boldsymbol{\xi} - \boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}\rangle\Big]\mathrm{d}V$$
$$= \int_V \langle \boldsymbol{a}'_\delta, \boldsymbol{b}\rangle\mathrm{d}V + \int_{S_{\mathrm{f}}} \langle \boldsymbol{a}'_\delta, \boldsymbol{s}\rangle\mathrm{d}S_{\mathrm{f}} + \int_{S_{\mathrm{c}}} \Big\{\langle \boldsymbol{a}'_\delta, \boldsymbol{s}_{\mathrm{c}}\rangle + \big\langle \delta(\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{s}_{\mathrm{c}}), \mathrm{ax}\log\boldsymbol{H} - \mathrm{ax}\log\boldsymbol{H}_{\mathrm{c}}\big\rangle\Big\}\mathrm{d}S_{\mathrm{c}}, \tag{71}$$

and

$$\int_V \delta\left[\langle \hat{\boldsymbol{S}}, \boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}\rangle - v\right]\mathrm{d}V \\ = \int_V \langle \boldsymbol{a}'_\delta, \boldsymbol{b}\rangle \mathrm{d}V + \int_{S_\mathrm{f}} \langle \boldsymbol{a}'_\delta, \boldsymbol{s}\rangle \mathrm{d}S_\mathrm{f} + \int_{S_\mathrm{c}} \left\{\langle \boldsymbol{a}'_\delta, \boldsymbol{s}_\mathrm{c}\rangle + \langle \delta(\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{s}_\mathrm{c}), \mathrm{ax}\log\boldsymbol{H} - \mathrm{ax}\log\boldsymbol{H}_\mathrm{c}\rangle\right\}\mathrm{d}S_\mathrm{c}. \tag{72}$$

Eqs. (71) and (72) are the most general *Three-field* and *Two-field Principles*, respectively, in finite elasticity mechanics of a polar medium in the context of the helicoidal modeling, with the compatibility at the constrained surface accounted for as a weak condition. They correspond respectively to the well-known Principle of Hu–Washizu (Washizu, 1968) and Principle of Hellinger–Reissner (Reissner, 1953), and will inherit the names. The relevant Euler–Lagrange equations are recognizable in the scalar Eqs. (64) and (70). The unknowns of the three-field formulation are $\boldsymbol{A}'$, $\boldsymbol{\xi}$ and $\hat{\boldsymbol{S}}$, and the constraint reaction $\boldsymbol{s}_\mathrm{c}$ as a boundary unknown; the same unknowns except $\boldsymbol{\xi}$ pertain to the two-field formulation.

From the three-field principle, Eq. (71), by forcing as identity the internal compatibility, $\boldsymbol{\xi} \equiv \boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}$, and by assuming implicitly $\hat{\boldsymbol{S}} \equiv w_{/\xi}$, one obtains the *One-field Principle*

$$\int_V \delta w\,\mathrm{d}V = \int_V \langle \boldsymbol{a}'_\delta, \boldsymbol{b}\rangle \mathrm{d}V + \int_{S_\mathrm{f}} \langle \boldsymbol{a}'_\delta, \boldsymbol{s}\rangle \mathrm{d}S_\mathrm{f} + \int_{S_\mathrm{c}} \{\langle \boldsymbol{a}'_\delta, \boldsymbol{s}_\mathrm{c}\rangle + \langle \delta(\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{s}_\mathrm{c}), \mathrm{ax}\log\boldsymbol{H} - \mathrm{ax}\log\boldsymbol{H}_\mathrm{c}\rangle\}\mathrm{d}S_\mathrm{c}. \tag{73}$$

The relevant Euler–Lagrange equation is $(\boldsymbol{HX}w_{/\xi})_{/\bullet} + \boldsymbol{b} = \boldsymbol{0}$, and the natural boundary conditions are $\boldsymbol{HX}w_{/\xi}\boldsymbol{v} = \boldsymbol{s}$ at S_f, and $\boldsymbol{HX}w_{/\xi}\boldsymbol{v} = \boldsymbol{s}_\mathrm{c}$ and $\log\boldsymbol{H} = \log\boldsymbol{H}_\mathrm{c}$ at S_c. The unknowns reduce to just $\boldsymbol{A}'$, and $\boldsymbol{s}_\mathrm{c}$. By forcing as identity the boundary compatibility $\log\boldsymbol{H} \equiv \log\boldsymbol{H}_\mathrm{c}$ as well, the classical *Principle of Virtual Work* $\int_V \delta w\,\mathrm{d}V = \int_V \langle \boldsymbol{a}'_\delta, \boldsymbol{b}\rangle \mathrm{d}V + \int_{S_\mathrm{f}} \langle \boldsymbol{a}'_\delta, \boldsymbol{s}\rangle \mathrm{d}S_\mathrm{f}$ is obtained from Eq. (73).

We close this section with a remark about the treatment of selective constraints in the context of the helicoidal modeling. Often, in structural analysis, we are concerned with problems where material points are constrained along some directions while they are free along other directions (we say that the constraint is selective). In such cases, it is expedient to decompose the scalar products into the sum of works done by the single pairs of working components, and associate each of them with the appropriate region, either S_f or S_c. This is a standard practice in the classical modeling based on variables pertaining to the Euclidean space, but it can be a difficult task in this context, due to the inherent non-linearity of the parameterization of motion of rototranslating particles. As a simple example, consider the rototranslation of a boundary particle which is constrained (c) in position and free (f) to rotate, i.e. $\boldsymbol{H}_\mathrm{b} = \boldsymbol{A}'_\mathrm{b}\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{X}'_\mathrm{c}\boldsymbol{\alpha}'_\mathrm{f}\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{X}^{\mathrm{T}} = \boldsymbol{X}'_\mathrm{c}\boldsymbol{\Phi}_\mathrm{f}\boldsymbol{X}^{\mathrm{T}}$. In such case, the compatibility term in Eqs. (67) or (69) writes $\langle \delta(\boldsymbol{\Lambda X}\hat{\boldsymbol{S}}\boldsymbol{v}), \mathrm{ax}\log(\boldsymbol{X}'\boldsymbol{\Phi}\boldsymbol{X}^{\mathrm{T}}) - \mathrm{ax}\log(\boldsymbol{X}'_\mathrm{c}\boldsymbol{\Phi}_\mathrm{f}\boldsymbol{X}^{\mathrm{T}})\rangle$, and splitting it into the sum of scalar conditions of the two kinds is clearly a subtle matter. When linearizing the variational principles for a practical solution, we have to tackle this decomposition in case of selective constraints. That is the reason why we will not consider the linearization of the constraint term in this paper, but we will delay it for a future work.

5.6. Consistent linearization

The variational principles stated above are of the form $\Pi_\delta = 0$, with Π_δ the appropriate non-linear virtual functional depending on the unknown fields and (linearly) on the relevant virtual fields. The linearization of the virtual functional is a standard process in computational mechanics; a boundary-value problem, for instance, is iteratively solved by the Newton–Raphson method on the incremental form $\Pi_\delta + \partial\Pi_\delta = 0$ of the variational principle, obtained from a truncated Taylor expansion. On a finite element approximation, as it is well known, the functional Π_δ and its variation $\partial\Pi_\delta$ give respectively the right-hand-side and the coefficients (the tangent matrix) of the resolving set of linear equations.

Due to the inherent non-linearity of the oriento-position field $\boldsymbol{A}'$, the linearization of the virtual functional is a quite difficult task. As a matter of fact, the variation $\partial\Pi_\delta$ is expected to depend on the unknown fields and (linearly) on the relevant virtual fields, on appropriate incremental fields, and in case on mixed virtual-incremental fields, as forewarned by the differentiation formulae for rototranslations, Eqs. (27). It is our belief that incremental fields consistent with the virtual ones should be conveniently exploited: in particular, the differential dual vector $\boldsymbol{a}'_\partial$ is here taken as the oriento-position incremental field. Note that of course, this choice demands for a consistent updating mechanism in the solution process—namely a multiplicative updating, as it will be exploited in Part III of this work.

The most general, Three-field Principle of Hu–Washizu, Eq. (71), is linearized under the simplifying hypothesis of constant density field for the self-based external loads $\boldsymbol{X}'^{\mathrm{T}}\boldsymbol{b}$ and $\boldsymbol{X}'^{\mathrm{T}}\boldsymbol{s}$. Moreover, the linearization of the contribution $\Pi_{\delta S_\mathrm{c}}$ from the constrained boundary is not attempted here. Derivation flows plain using the kinematical relations given far above and the elasticity formulae Eqs. (57)–(61). Details are found in Merlini and Morandini (2002) and just the results are given here. We collect the virtual functional and its variation,

$$\begin{aligned}\Pi_\delta &= \int_V \{\langle \boldsymbol{A}'\delta(\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{k}'), \tilde{\boldsymbol{S}}\rangle - \langle \delta\hat{\boldsymbol{S}}, \boldsymbol{\xi} - \boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}\rangle\}\mathrm{d}V - \int_V \langle \boldsymbol{X}'^{\mathrm{T}}\boldsymbol{a}'_\delta, \boldsymbol{X}'^{\mathrm{T}}\boldsymbol{b}\rangle \mathrm{d}V - \int_{S_\mathrm{f}} \langle \boldsymbol{X}'^{\mathrm{T}}\boldsymbol{a}'_\delta, \boldsymbol{X}'^{\mathrm{T}}\boldsymbol{s}\rangle \mathrm{d}S_\mathrm{f} + \Pi_{\delta S_\mathrm{c}},\\ \partial\Pi_\delta &= \partial\Pi_{\delta E} + \partial\Pi_{\delta G} + \partial\Pi_{\delta C} + \partial\Pi_{\delta S_\mathrm{c}},\end{aligned} \tag{74}$$

with

$$\begin{aligned}\partial\Pi_{\delta E} &= \int_V \{\delta\boldsymbol{\xi}\}^{\mathrm{T}} : [\mathbb{E}] : \{\partial\boldsymbol{\xi}\}\mathrm{d}V,\\ \partial\Pi_{\delta G} &= \int_V \langle \boldsymbol{A}'\partial\delta(\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{k}'), \tilde{\boldsymbol{S}}\rangle \mathrm{d}V - \int_V \langle \partial(\boldsymbol{X}'^{\mathrm{T}}\boldsymbol{a}'_\delta), \boldsymbol{X}'^{\mathrm{T}}\boldsymbol{b}\rangle \mathrm{d}V - \int_{S_\mathrm{f}} \langle \partial(\boldsymbol{X}'^{\mathrm{T}}\boldsymbol{a}'_\delta), \boldsymbol{X}'^{\mathrm{T}}\boldsymbol{s}\rangle \mathrm{d}S_\mathrm{f},\\ \partial\Pi_{\delta C} &= -\int_V \{\langle \delta\hat{\boldsymbol{S}}, \partial\boldsymbol{\xi} - \boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{A}'\partial(\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{k}')\rangle + \langle \partial\hat{\boldsymbol{S}}, \delta\boldsymbol{\xi} - \boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{A}'\delta(\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{k}')\rangle + \langle \partial\delta\hat{\boldsymbol{S}}, \boldsymbol{\xi} - \boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}\rangle\}\mathrm{d}V,\end{aligned} \tag{75}$$

the three main contributions to the linearized virtual functional, respectively the *elastic* contribution, the *geometric* contribution due to pre-stress and external loads, and the *constraint* contribution accounting for internal compatibility. The nucleuses of the elastic and geometric tangent matrices can be recognized in Eqs. $(75)_{1,2}$, respectively. In particular, it is noted that the geometric stiffness due to pre-stress ensues symmetrical, while the same may not be true in general for the geometric term due to external loads.

The linearized virtual functional depends on the parameters describing the current oriento-position $\boldsymbol{A}'$ and on the strain and stress parameters $\boldsymbol{\xi}$ and $\hat{\boldsymbol{S}}$ as free variables ($\tilde{\boldsymbol{S}}$ is a short for $\boldsymbol{H}\boldsymbol{X}\hat{\boldsymbol{S}}$). Besides, it depends on the relevant virtual, incremental and mixed virtual-incremental variables, namely the differential oriento-position vectors and curvature tensors $\boldsymbol{a}'_\delta$, $\boldsymbol{a}'_\partial$, $\boldsymbol{a}'_{\partial\delta}$, $\boldsymbol{k}'_\delta$, $\boldsymbol{k}'_\partial$, $\boldsymbol{k}'_{\partial\delta}$ (recall Eqs. (42)), and the strain and stress variations $\delta\boldsymbol{\xi}$, $\partial\boldsymbol{\xi}$, $\delta\hat{\boldsymbol{S}}$, $\partial\hat{\boldsymbol{S}}$, $\partial\delta\hat{\boldsymbol{S}}$. The mixed virtual-incremental variables are in general not null in non-linear mechanics. This happens of course for the mixed differentials $\boldsymbol{a}'_{\partial\delta}$ and $\boldsymbol{k}'_{\partial\delta}$, but this will also be true for any other mixed variations in a finite-element context, unless the field variables are assumed linearly dependent on the discrete set of global variables.

Linearization of the One-field Principle Eq. (73) yields the following obvious results,

$$\begin{aligned}\Pi_\delta &= \int_V \langle \boldsymbol{A}'\delta(\boldsymbol{A}'^{T}\boldsymbol{k}'), \tilde{\boldsymbol{S}}\rangle \mathrm{d}V - \int_V \langle \boldsymbol{X}'^{\mathrm{T}}\boldsymbol{a}'_\delta, \boldsymbol{X}'^{\mathrm{T}}\boldsymbol{b}\rangle \mathrm{d}V - \int_{S_\mathrm{f}} \langle \boldsymbol{X}'^{\mathrm{T}}\boldsymbol{a}'_\delta, \boldsymbol{X}'^{\mathrm{T}}\boldsymbol{s}\rangle \mathrm{d}S_\mathrm{f} + \Pi_{\delta S_\mathrm{c}},\\ \partial\Pi_\delta &= \partial\Pi_{\delta E} + \partial\Pi_{\delta G} + \partial\Pi_{\delta S_\mathrm{c}},\end{aligned} \tag{76}$$

with now

$$\partial\Pi_{\delta E} = \int_V \{\boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{A}'\delta(\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{k}')\}^{\mathrm{T}} : [\mathbb{E}] : \{\boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{A}'\partial(\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{k}')\}\mathrm{d}V, \tag{77}$$

and $\partial\Pi_{\delta G}$ the same as in Eq. $(75)_2$. The free variables reduce to the parameters of $\boldsymbol{A}'$, and the variation variables to just the kinematical differentials ($\tilde{\boldsymbol{S}}$ is now a short for $\boldsymbol{H}\boldsymbol{X}w_{/\xi}$). Eqs. (76) and (77) cover also the case of the classical Principle of Virtual Work once the contributions $\Pi_{\delta S_c}$ and $\partial\Pi_{\delta S_c}$ from the constrained surface are discarded.

6. Conclusion

In this paper, well-known variational principles in solid mechanics have been presented and discussed for polar finite elasticity in the new light of the helicoidal modeling. Crucial to this explanation appear a number of original contributions.

1. The introduction of dual numbers in the polar description of deformable continua, with the identification of pole-independent scalar invariants as the magnitude of a dual vector and the determinant of a dual tensor, and with a workable formulation for the scalar product between covariant and contravariant dual vectors and tensors.
2. The differentiation up to third-order of rotation and rototranslation tensors and the identification of the relevant characteristic differential vectors.
3. The concept of the helicoidal modeling itself, closely related to the tangent space of the kinematic field of oriento-positions and inheriting, in a unifying description, the properties peculiar of the angular kinematics. In particular, the characterization of the spatial, virtual and incremental variations of the oriento-position field by means of mixed differential vectors and tensors.
4. The identification of the strain and stress dual parameters for a polar hyperelastic medium.
5. The identification of a consistent dual multiplier for the boundary compatibility condition.
6. The linearized form of the virtual functionals explicitly depending on the virtual, incremental and mixed virtual-incremental variation variables.

The proposed formulation is of course liable to improvement. In particular, we envisage the extension of the polar elasticity to embrace non-hyperelastic models and then the introduction of constitutive laws for plasticity. Besides, we expect a careful investigation of the weak form of the external constraints and its linearization. In fact, forcing the external constraints at the boundary surface in real analyses can be quite difficult in the context of the helicoidal modeling, whenever selective constraints have to be considered.

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